Buckling of a heavy tapered rod

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Abstract

The study of buckling of a tapered rod leads to the nonlinear eigenvalue problem

$$\{A(s)u'(s)\}' + \mu \sin u(s) = 0 \text{ for all } s \in (0,1), \tag{1}$$

$$u(1) = \lim_{s \to 0} A(s)u'(s) = 0 \tag{2}$$

$$\text{and} \qquad \int_0^1 A(s)u'(s)^2 ds < \infty \tag{3}$$

where $A \in C([0,1])$ is such that $A(s) > 0$ for all $s > 0$ and $\lim_{s\to 0} A(s)/s^p = L$ for some constants $p \geq 0$ and $L \in (0, \infty)$. There is a number $\Lambda(A) \geq 0$ such that, for $\mu \leq \Lambda(A)$, $u \equiv 0$ is the only solution of the problem and it minimizes the

energy in the space of all admissible configurations. For $\mu > \Lambda(A)$, the energy is minimized by a non-trivial solution. For $p = 0$, this is a well understood classical problem studied by D. Bernoulli and Euler. For $0 < p < 2$, the problem is singular but its bifurcation diagram remains similar to the case $p = 0$. At $p = 2$, striking changes occur.

(1) For $0 \leq p < 2$, $\lim_{s\to 0} u(s) \in (-\pi, \pi)$ for all non-trivial solutions whereas $\lim_{s\to 0} u(s) = \pm\pi$ if $p \geq 2$.

(2) For $0 \leq p \leq 2, \Lambda(A) > 0$ whereas $\Lambda(A) = 0$ for $p > 2$.

(3) For $0 \leq p < 2$, bifurcation from the solution $u \equiv 0$ occurs only at a discrete set of eigenvalues μ_i where $\mu_1 = \Lambda(A)$ and $\lim_{i\to\infty} \mu_i = \infty$. For $p = 2$, there is a number $\Lambda_e(A) \in [\Lambda(A), \infty)$ such that bifurcation occurs at every value $\mu \in [\Lambda_e(A), \infty)$.

The properties of the linearized problem, in which (1) is replaced by

$$\{A(s)u'(s)\}' + \mu u(s) = 0 \text{ for all } s \in (0, 1), \tag{4}$$

also change at $p = 2$. For $0 \leq p < 2$, its spectrum $\{\mu_i\}$is discrete and all the eigenfunctions have only a finite number of zeros in $[0, 1]$. For $p = 2, \Lambda_e(A)$ belongs to the essential spectrum and there may be no eigenfunctions. Furthermore, for $p = 2$ and $\mu > \Lambda_e(A)$ all solutions of (4) have infinitely many zeros in $[0, 1]$, but solutions of the nonlinear problem have only a finite number of zeros.

1 Introduction

In this paper we discuss one of the simplest models for the planar buckling of a tapered column. We consider two different situations in both of which one end of the column is clamped and the other is free. In the first problem, no load is applied to the column and we are concerned with buckling due to the column's own weight under gravity which acts in the direction of clamping. In the second one, we neglect gravity and consider buckling due to a compressive force applied at the free end, parallel to the direction of clamping at the other end. In the situation we treat the mathematical formulation of the both problems can be reduced to the same form by a change of variable. Hence for the rest of this introduction we discuss only the case of a column buckling under its own weight. The problem of a column buckling under a compressive force is formulated and discussed in Section 7.

1.1 A tapered elastic rod

We consider an unshearable, inextensible rod whose resistance to bending is governed by the classical Bernoulli - Euler law for the elastica. It has one free end, the other being clamped vertically upwards with gravity acting vertically downwards. The rod is made of a homogeneous material of constant density $\rho > 0$, but the cross-sectional area can vary along its length. The configuration in which the rod is straight is in equilibrium but may be unstable since a buckled state may have lower energy. In seeking to obtain a tall column which remains straight from a given quantity of material, it is natural to consider situations where the cross-sections diminish near the free end.

In order to analyse the problem we introduce a standard formulation which we now present in an informal way. However, it is easy to give it a rigorous expression as an example in the theory of planar configurations of special Cosserat rods by following closely the development in Chapter IV.1 of Antman's treatise [2]. See also Chapter VIII.15 and 16 of [2] for further discussion of the constitutive assumptions. We begin by describing a three dimensional region which will be used to establish a reference configuration for a rod. Let B be an open bounded subset of $\mathbb{R}^3$ whose sections

$$D(z) = \{(x,y) : (x,y,z) \in B\}$$

have the following properties.

(i) $D(z) \neq \emptyset \qquad \Longleftrightarrow \qquad 0 < z < 1.$

(ii) $D(z)$ is simply connected and $(x,y) \in D(z) \Longleftrightarrow (x,-y) \in D(z)$.

(iii) $(0,0)$ is the centroid of $D(z)$, that is $\iint_{D(z)} x dx dy = 0$

(iv) There are functions S and $I \in C([0,1])$ such that, for $0 < z < 1$, $S(z)$ is the area of $D(z)$ and $I(z)$ is its moment of inertial about the $y-$axis. Thus $S(z) = \iint_{D(z)} dxdy > 0$ and $I(z) = \iint_{D(z)} x^2 dxdy > 0$ for all $z \in (0,1)$ and we suppose, in addition, that $S(0) > 0$ and $I(0) > 0$.

We think of B as being occupied by a rod-like body in its reference configuration. A planar configuration of the rod is identified with a curve in the $(x,z)-$plane which will be taken to be formed by the centroids of these sections. Suppose that the inextensible rod has unit length and consider a smooth planar configuration. We use arc-length, s, measured from the free end, $r(0)$, for a parametric representation, $r : [0,1] \to \mathbb{R}^2$, of this configuration. Then there is a unique angle, $\theta(s) \in [0, 2\pi)$, such that

$$r'(s) = -\left(\sin\theta(s), \cos\theta(s)\right). \tag{1.1}$$

Choosing axes such that gravity acts in the direction of $(0,-1), \theta(s)$ measures the angle between the tangent to the rod at position $r(s)$ and the gravitational force. See Figure 1.

Fig. 1 Notation and conventions

Choosing the origin of the coordinates so that $r(1) = (0,0)$, the configuration of the rod is recovered from the angle θ via

$$r(s) = \left(\int_s^1 \sin\theta(t)dt, \int_s^1 \cos\theta(t)dt\right) \text{ for } 0 \leq s \leq 1. \tag{1.2}$$

Thus the reference configuration is given by $\theta(s) \equiv 0$ or $r(s) = (0, 1-s)$.

Using $M(s)$ to denote the bending moment at $r(s)$, the equilibrium conditions are expressed by the differential equation

$$M'(s) + \rho g[\int_0^s S(1-\tau)d\tau]\sin\theta(s) = 0 \text{ for } 0 < s < 1 \tag{1.3}$$

where $g > 0$ is the gravitational constant and $S(z)$ is the area of the horizontal section $D(z)$ at height z in the reference configuration. Since the end $r(0)$ is free whereas the other end $r(1)$ is clamped vertically upwards, we must impose the boundary conditions

$$\lim_{s\to 0} M(s) = 0 \text{ and } \theta(1) = 0. \tag{1.4}$$

Finally the Bernoulli - Euler constitutive relation for the elastica is expressed as

$$M(s) = EI(1-s)\theta'(s) \tag{1.5}$$

where $E > 0$ is a material constant and $I(z)$ is the moment of inertia of the horizontal section at height z in the reference configuration about the axis through $(0,0,z)$ and perpendicular to the $(x,z)-$plane. See (16.12) of [2]. The equilibrium equation (1.3) becomes

$$\{I(1-s)\theta'(s)\}' + \xi[\int_0^s S(1-\tau)d\tau]\sin\theta(s) = 0 \text{ for } 0 < s < 1 \tag{1.6}$$

where $\xi = \frac{\rho g}{E} > 0$ and $I, S \in C([0,1])$ are given functions. We seek solutions θ satisfying the boundary conditions (1.4) which become

$$\lim_{s\to 0} I(1-s)\theta'(s) = 0 \text{ and } \theta(1) = 0. \tag{1.7}$$

Motivated by the work on the shape of the tallest column,[19] and [8], we are particularly interested in cases where $S(1-s)$ and $I(1-s) \to 0$ as $s \to 0$, so the first boundary condition in (1.7) cannot be replaced by $\lim_{s\to 0}\theta'(s) = 0$. For the same reason, configurations which satisfy the boundary conditions (1.7) do not necessarily have finite elastic energy so this has to be ensured separately. The total energy of the configuration (1.2) is given by

$$\int_0^1 \frac{1}{2}EI(1-s)\theta'(s)^2 - \rho g[\int_0^s S(1-\tau)d\tau]\{1-\cos\theta(s)\}\,ds \tag{1.8}$$

Since $S \in C([0,1])$, this energy is finite if and only if

$$\int_0^1 I(1-s)\theta'(s)^2 ds < \infty. \tag{1.9}$$

We can now give a precise statement of the mathematical problem to be discussed. Given a constant $\xi > 0$ and functions I and $S \in C([0,1])$ with $I(z)$ and $S(z) > 0$ for $z < 1$, we seek solutions of (1.6) which satisfy (1.7) and (1.9). The following change of variables brings this problem into a more convenient form which coincides with the problem of a loaded rod discussed in Section 7. Given a function $S \in C([0,1])$ with $S(z) > 0$ for $z < 1$, let

$$Z = \int_0^1 \int_0^\sigma S(1-\tau)d\tau d\sigma \text{ and } t = t(s) = \frac{1}{Z}\int_0^s \int_0^\sigma S(1-\tau)d\tau d\sigma. \tag{1.10}$$

Then, set

$$u(t) = \theta(s) \text{ and } A(t) = I(1-s)\int_0^s S(1-\tau)d\tau \text{ where } t = t(s). \tag{1.11}$$

Clearly t increases from 0 to 1 as s increases from 0 to 1. Furthermore,

$$\int_0^1 I(1-s)\theta'(s)^2 ds = \frac{1}{Z}\int_0^1 A(t)u'(t)^2 dt,$$
$$A(t)u'(t) = ZI(1-s)\theta'(s) \text{ where } t = t(s),$$

the equation (1.6) becomes

$$\{A(t)u'(t)\}' + \mu \sin u(t) = 0 \text{ for } 0 < t < 1$$

where $\mu = Z^2\xi$ and the energy (1.8) becomes

$$\frac{E}{Z}\int_0^1 \frac{1}{2}A(t)u'(t)^2 - \mu\{1-\cos u(t)\}\,dt$$

With this in mind we introduce the following terminology.

Definition 1.1 *A profile for a column with tapering of order $p \geq 0$ is a function $A \in C([0,1])$ such that $A(t) > 0$ for $0 < t \leq 1$ and there exists $L \in (0,\infty)$ such that $\lim_{t\to 0}\frac{A(t)}{t^p} = L$.*

For such a profile there exist constants $K_1 \geq K_2 > 0$ such that

$$K_2 t^p \leq A(t) \leq K_1 t^p \text{ for all } t \in [0,1]. \tag{1.12}$$

We now give the formal statement of the mathematical problem to be considered. Consider a profile A with tapering of order $p \geq 0$ and a constant $\mu > 0$.

Definition 1.2 *A solution of Problem P is a function $u \in C^1((0,1])$ such that $Au' \in C^1((0,1])$,*

$$\{A(t)u'(t)\}' + \mu \sin u(t) = 0 \textit{ for all } t \in (0,1], \tag{1.13}$$
$$u(1) = \lim_{t\to 0} A(t)u'(t) = 0 \tag{1.14}$$
$$\textit{and} \qquad \int_0^1 A(t)u'(t)^2 dt < \infty. \tag{1.15}$$

In fact, for a given profile we would like to study the solutions of Problem P as the parameter μ varies. Our results show that in several respects (shape of the buckled configurations, nature of the bifurcation diagrams) tapering of order 2 plays a critical role, in the sense that the situation when $p < 2$ is very different from what occurs when $p \geq 2$.

Remark 1 As we show in Section 7, Problem P with $A(t) = I(1-t)$ and $\mu = \frac{f}{E}$ arises directly as the model for a tapered Euler rod whose cross-sections satisfy the conditions (i) to (iv) with a force $F = f(0,-1)$ applied to its free end when gravity is neglected.

Remark 2 To interpret our results concerning Problem P in terms of a rod buckling under its own weight, note that if

$$\lim_{s\to 0}\frac{S(1-s)}{s^q}=K>0 \text{ and } \lim_{s\to 0}\frac{I(1-s)}{s^r}=J>0 \tag{1.16}$$

then

$$\lim_{t\to 0}\frac{A(t)}{t^p}=L \qquad \text{where } p=\frac{r+q+1}{q+2} \text{ and } L=\frac{JK}{q+1}\left\{\frac{Z(q+1)(q+2)}{K}\right\}^p.$$

In particular, in the case of a uniform column where S and I are constant, $q=r=0$ and hence $p=\frac{1}{2}$.

Remark 3 In the linear theory of tapered columns buckling under their own weight, [14], [19], [8], it is often assumed that the sections are all similar since in this case I is proportional to S^2. More precisely, in addition to the assumptions (i) to (iv), we suppose that the sections $D(z)$ have the following property.

(v) There are a function $\alpha\in C([0,1])$ with $\alpha(z)>0$ for $z<1$ and a set $D\subset\mathbb{R}^2$ such that $D(z)=\alpha(z)D$ for all $z\in(0,1)$.

Then

$$S(z)=\alpha(z)^2\,|D| \text{ where } |D| \text{ is the area of } D$$

and

$$I(z)=CS(z)^2 \text{ where } C=\frac{1}{|D|^2}\iint_D x^2dxdy.$$

Under these conditions, $r=2q$ in (1.16),

$$A(t)=CS(1-s)^2\int_0^s S(1-\tau)d\tau$$

and, if

$$\lim_{s\to 0}\frac{S(1-s)}{s^q}=K>0,$$

then

$$\lim_{t\to 0}\frac{A(t)}{t^p}=L \qquad \text{where } p=\frac{3q+1}{q+2} \text{ and } L=CZ^pK^{3-p}(q+1)^{p-1}(q+2)^p.$$

Our results show that the case $p=2$ plays a critical role and this corresponds to $\lim_{s\to 0}\frac{S(s)}{s^3}=K\in(0,\infty)$ where $L=100CZ^2K$, in the case of a column with geometrically similar cross-sections. In the case where $S(z)$ is constant (equivalently $A(t)=t^{1/2}$) the equation (1.6) was derived by Daniel Bernoulli [4] in the same paper as his original proposition of the Bernoulli - Euler law for the bending moment. See equation (90) in Truesdell's authoritative commentaries [28].

1.2 Summary of the results

Our results concerning Problem P are given in Sections 3 to 5 after a number of essential preliminary technical issues have been settled in Section 2.

Clearly, for a given rod, the energy is proportional to

$$J_\mu(\theta) = \int_0^1 \frac{1}{2}A(t)u'(t)^2 - \mu\{1-\cos\theta(t)\}\,dt \qquad (1.17)$$

and solutions of Problem P correspond to stationary points of J_μ on an appropriate space. Indeed, in Section 2 we introduce the space H_A of all functions $\theta : [0,1] \to \mathbb{R}$ associated with configurations which are clamped vertically upwards at the origin and have finite energy. (In this setting, the condition at the free end appears as a natural boundary condition satisfied by all stationary points of J_μ.) It is a Hilbert space with scalar product

$$\langle u, v\rangle_A = \int_0^1 A(t)u'(t)v'(t)dt. \qquad (1.18)$$

For $p \geq 1$, the elements of H_A are not necessarily bounded as $s \to 0$ and so the functional J_μ is not Fréchet differentiable on H_A for profiles A having tapering of order p with p large. The main properties of the space H_A are established in Section 2 and enable us to investigate the smoothness and compactness properties of J_μ on H_A. It turns out that for any profile, J_μ is differentiable at $u \in H_A$ with respect to directions v in a dense subspace E_A of H_A, in the sense made precise in Lemma 2.10. Furthermore, u is a solution of Problem P if and only if $\frac{d}{dx}J_\mu(u+xv)\mid_{x=0}= 0$ for all $v \in E_A$, as is shown in Theorem 3.1. It follows easily that for any profile A and any $\mu > 0$, the Problem P has a unique (up to sign) solution u_μ having minimum energy and u_μ depends continuously on μ in H_A. In Theorems 3.3 to 3.9 we show that there is a constant $\Lambda(A) \geq 0$ such that $u_\mu > 0$ on $(0,1)$ if $\mu > \Lambda(A)$ and that $u_\mu \equiv 0$ on $(0,1)$ if $0 < \mu \leq \Lambda(A)$.

A first indication that tapering of order 2 is critical is the observation that $\Lambda(A) > 0$ if A is a profile with tapering of order $p \leq 2$ whereas $\Lambda(A) = 0$ if $p > 2$. Thus, for a rod whose profile has tapering of an order greater than 2, every stable equilibrium configuration is buckled no matter how small μ is. Furthermore, the shapes of the stable buckled configurations also change dramatically at $p = 2$. In all cases the angle $u_\mu(s)$ decreases monotonely from the free end to the clamped end, but for $p \geq 2$, the free end of the rod always points vertically downwards (i.e. $u_\mu(0) = \pi$), whereas for $p < 2$, $u_\mu(0) < \pi$ and, in fact, $u_\mu(0) \to 0$ as $\mu \to \Lambda(A)$ from above. See Figures 2 to 6. This behaviour can be reformulated in terms of the bifurcation of the branch of stable equilibria. For the H_A-norm the curve u_μ bifurcates from the vertical configuration $u \equiv 0$ at $\mu = \Lambda(A)$. This remains true for the $L^\infty(0,1)-$norm provided that $p < 2$, but when $p = 2$ there is a discontinuity at $\mu = \Lambda(A)$ since, in this case, $u_\mu \equiv 0$ for $0 < \mu \leq \Lambda(A)$ and $\|u_\mu\|_{L^\infty(0,1)} = \pi$ for all $\mu > \Lambda(A)$. Moreover, there is a strong boundary layer phenomenon as μ approaches $\Lambda(A)$ from above when $p = 2$ because $u_\mu(t) \to 0$ uniformly on compact subsets of $(0,1]$, but $\lim_{t\to 0} u_\mu(t) = \pi$ for all $\mu > \Lambda(A)$. These statements are justified by Theorems 3.9 to 3.13.

The following figures show some equilibrium configurations for rods with various orders of tapering p and values of the parameter μ. The graphs of the corresponding solutions u_μ of Problem P are shown in Section 4. The configurations below are obtained from u_μ using (1.2) and so they apply to the case of a loaded rod in the absence of gravity described in Section 7. The configurations for rods buckling under their own weight are rescaled versions of those shown below obtained by combining (1.2) with the change of variables (1.10),(1.11).

Fig. 2 $p = 1/3$ and $\mu = 2$

Fig. 3 $p = 1/2$ and $\mu = 4$

Fig. 4 $p = 2$ and $\mu = 2$

Fig. 5 $p = 2$ and $\mu = 50$

More information about the properties (monotonicity etc.) of the stable equilibria are contained in Theorems 3.3 and 3.10 and Corollaries 3.4 and 4.5, but we now pass on to a discussion of the complete bifurcation diagrams. For this the linearization of Problem P becomes important.

Definition 1.3 *A solution of Problem PL is a function $u \in C^1((0,1])$ such that $Au' \in C^1((0,1])$,*

$$\{A(t)u'(t)\}' + \mu u(t) = 0 \textit{ for all } t \in (0,1], \tag{1.19}$$

and (1.14) and (1.15) are satisfied. If $u \not\equiv 0$, it is called an eigenfunction associated with the eigenvalue μ.

The spectral theory of this linear boundary value problem has been developed in [24] and it summarized in Section 2. The bifurcation point $\Lambda(A)$ for the stable equilibria of the nonlinear problem is characterized as the infimum of the associated Rayleigh quotient Q_A for Problem PL and some estimates for $\Lambda(A)$ are given. For $p \leq 2$, the non-trivial solutions of Problem PL are the eigenfunctions of a bounded positive self-adjoint operator $T : H_A \to H_A$ (defined in Proposition 2.3) and the spectrum of Problem PL is the spectrum of its inverse T^{-1}. In all cases, $\frac{1}{\Lambda(A)} = \max \sigma(T)$ where $\sigma(T)$ denotes the spectrum of T. For $p < 2$, T is compact and $\sigma(T)$ consists of 0 (which is not an eigenvalue) and a sequence of simple eigenvalues converging to 0. However, for $p = 2$, T is not compact and its essential spectrum has a positive maximum given by $\frac{4}{L}$ where $L = \lim_{t\to 0} A(t)/t^2$. Again for $p = 2$, the operator T may or may not have eigenvalues depending on the form of the profile A. We show in Section 6 that the presence of essential spectrum for Problem PL when $p = 2$ has a profound effect on the nature of the bifurcation diagram for Problem P. The degree of tapering also affects the nodal structure of solutions in a critical way. For $p < 2$, all non-trivial solutions of both Problem P and PL have only a finite number of zeros in $[0,1]$. This remains true for the nonlinear Problem P when $p = 2$, but for $p = 2$ and $\mu > \Lambda_e(A)$ all solutions of Problem PL have infinitely many zeros in $[0,1]$.

The precise nature of the relationship between the Problems P and PL, and consequently the bifurcation diagram for Problem P, depend on the smoothness of the functional $J_\mu : H_A \to \mathbb{R}$. For $p < 2$, $J_\mu \in C^2(H_A)$ and T is the Fréchet derivative of ∇J_μ at $u = 0$. Standard bifurcation results, local and global, can be applied to obtain a bifurcation diagram that closely resembles the well-known one for the case $A(t) \equiv 1$ which goes back to Euler himself. See Figure 7 and Theorems 4.3 and 4.6 for the justification. For $p = 2$, $J_\mu \in C^1(H_A)$ and ∇J_μ is still a compact perturbation of the identity, but ∇J_μ is not Fréchet differentiable even at $u = 0$. Nonetheless T is the Gâteaux derivative of ∇J_μ at $u = 0$ and every point in the interval $[\Lambda_e(A), \infty)$ is a bifurcation point for Problem P. In fact, for every $\mu > \Lambda_e(A)$, the Problem P has a sequence $\{u_k\}$ of distinct solutions such that $\|u_k\|_A \to 0$ as $k \to \infty$. Of course, if $\Lambda(A) < \Lambda_e(A)$ the point $\Lambda(A)$ is also a bifurcation point for Problem P. See Figures 8 and 9. Bifurcation diagrams of this type were first obtained by Benci and Fortunato, [3], and Bongers, Heinz and Küpper, [5], for the model problem

$$-\Delta u(x) + W(x)\,|u(x)|^\gamma\, u(x) = \lambda u(x) \text{ with } u \in H^1(\mathbb{R}^N)$$

where $\gamma > 0$ and $W(x) \geq C\,|x|^{\beta}$ for some $\beta > \frac{\gamma N}{2}$ and some positive constant C. See also [16], [17] and [26] for related work. The main tool for establishing these results is a variant of Ljusternik-Schnirelman theory based on the genus of a set. Our conclusions are given in Theorem 5.8. The following figures indicate the main features of the bifurcation diagrams.

Fig. 6 $p < 2$

Fig. 7 $p = 2$ and $\Lambda(A) < \Lambda_e(A)$

Fig 8 $p = 2$ and $\Lambda(A) = \Lambda_e(A)$

2 Preliminaries

We begin by introducing the space of all configurations of a rod with profile A which have finite energy. Up to equivalence of norms, this space depends only on the order of tapering.

2.1 The energy space H_A

Consider $p \in [0, \infty)$. If an element $u \in L^1_{loc}((0,1])$ admits a generalized derivative u' on $(0,1)$ such that $\int_0^1 s^p u'(s)^2 ds < \infty$, it follows that $u \in W^{1,1}((\varepsilon,1))$ for all $\varepsilon \in (0,1)$, and hence, from Théorème VIII.2 of [6], that (after modification on a set of measure zero) $u \in C((0,1])$.

For $p \geq 0$, let

$$H_p = \left\{ u \in L^1_{loc}((0,1]) : \int_0^1 s^p u'(s)^2 ds < \infty \text{ and } u(1) = 0 \right\}$$

with

$$\|u\|_p = \left\{ \int_0^1 s^p u'(s)^2 ds \right\}^{1/2}.$$

Clearly $\|\cdot\|_p$ is a norm on the linear space H_p and

$$u(x) = -\int_x^1 u'(s) ds$$

for all $u \in H_p$ and all $x \in (0,1]$. Hence, for $u \in H_p$ and $x \in (0,1]$,

$$|u(x)| \leq \|u\|_p \left\{ \int_x^1 s^{-p} ds \right\}^{1/2}$$

and so

$$|u(x)| \leq \|u\|_p \left\{ \frac{1 - x^{1-p}}{1-p} \right\}^{1/2} \quad \text{if } p \neq 1, \tag{2.1}$$

whereas

$$|u(x)| \leq \|u\|_p \left\{ \ln \frac{1}{x} \right\}^{1/2} \quad \text{if } p = 1. \tag{2.2}$$

Similarly, for $u \in H_p$ and $x, y \in (0,1]$,

$$|u(x) - u(y)| \leq \|u\|_p \left\{ \left| \int_x^y s^{-p} ds \right| \right\}^{1/2}. \tag{2.3}$$

Proposition 2.1 *(i) For $p \in [0,\infty)$, H_p with the scalar product*

$$\langle u, v \rangle_p = \int_0^1 s^p u'(s) v'(s) ds$$

is a Hilbert space.
(ii) For any bounded sequence $\{u_n\}$ in H_p there exist a function $u \in C((0,1])$ and a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \to u$ uniformly on $[\varepsilon, 1]$ for every $\varepsilon \in (0,1)$.
(iii) $H_p \cap L^\infty(0,1)$ is dense in H_p.
(iv) If $u \in H_p$ then so does $|u|$ and $|u|'(s)^2 = u'(s)^2$ almost everywhere on $(0,1)$.

Proof See [24].

Remark 1 If A is a profile for a column with tapering of order p, then

$$\langle u, v\rangle_A = \int_0^1 A(s)u'(s)v'(s)ds$$

is a scalar product on $H_A = H_p$ whose norm is equivalent to $\|\cdot\|_p$. Indeed,

$$\sqrt{K_2}\,\|u\|_p \leq \|u\|_A \leq \sqrt{K_1}\,\|u\|_p \tag{2.4}$$

with the constants given in (1.12). The Hilbert space $(H_A, \langle\cdot,\cdot\rangle_A)$ will be referred to as the **energy space for the profile A.** If the sequence $\{u_n\}$ converges weakly to u in H_A, then $u_n \to u$ uniformly on $[\varepsilon, 1]$ for every $\varepsilon \in (0,1)$.

Remark 2 Denoting by $AC_{loc}((0,1])$ the set of all functions that are absolutely continuous on $[\varepsilon, 1]$ for every $\varepsilon \in (0,1)$, the space H_A can be characterized as

$$\left\{u \in AC_{loc}((0,1]) : u(1) = 0 \text{ and } \int_0^1 A(s)u'(s)^2 ds < \infty\right\}.$$

Remark 3 Setting

$$u_\alpha(s) = s^\alpha(1-s) \text{ for } 0 < s \leq 1, \tag{2.5}$$

we see that $u_\alpha \in H_p \Longleftrightarrow \alpha > \frac{1-p}{2}$. Noting that the function $\ln\left\{\ln \frac{e}{s}\right\}$ belongs to H_1 and recalling (2.1),(2.2), we see that $H_p \subset L^\infty(0,1) \Longleftrightarrow p < 1$.

By (2.1)and (2.2), $H_p \subset L^2(0,1)$ for $p < 2$. The following result is in the spirit of Hardy's inequality, see 327 of [15], and shows that $H_p \subset L^2(0,1)$ for $p \leq 2$. For $p > 2$ and $\alpha \in (\frac{1-p}{2}, -\frac{1}{2}]$ the function u_α defined by (2.5) belongs to H_p but not to $L^2(0,1)$.

The usual scalar product and norm on $L^2(0,1)$ will be denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|_2$ respectively.

Lemma 2.2 *Let $0 \leq p \leq 2$. Then $H_p \subset L^2(0,1)$ and*

$$\left\{\int_0^1 u(s)^2 ds\right\}^{1/2} \leq 2\,\|u\|_p \tag{2.6}$$

for all $u \in H_p$.

Proof See [24].

2.2 The linearized problem

In this part we summarize the spectral theory of the linearized problem in which (1.13) is replaced by (1.19). We begin by introducing a bounded linear operator $T : H_A \to H_A$ associated with this linear boundary value problem. All of the results stated below are proved in [24].

Proposition 2.3 *Let A be a profile with tapering of order $p \in [0,2]$. There is a unique bounded linear operator $T : H_A \to H_A$ such that*

$$\langle T(u), v\rangle_A = \langle u, v\rangle \text{ for all } u, v \in H_A \tag{2.7}$$

where $\langle\cdot,\cdot\rangle$ is the usual scalar product on $L^2(0,1)$. Furthermore T is a positive self-adjoint operator in H_A and 0 is not an eigenvalue of T.

For $p < 2, T : H_A \to H_A$ is also compact.

The spectrum of T is the set

$$\sigma(T) = \{\lambda \in \mathbb{R} : T - \lambda I : H_A \to H_A \text{ is not an isomorphism}\}.$$

Recall (Theorem 1.6 of [11], for example) also that the discrete spectrum of T is the set

$$\sigma_d(T) = \{\lambda \in \sigma(T) : T - \lambda I : H_A \to H_A \text{ is a Fredholm operator}\}$$

and the essential spectrum is its complement

$$\sigma_e(T) = \sigma(T) \backslash \sigma_d(T).$$

It can be shown that $\sigma_d(T)$ is formed by the isolated eigenvalues of T which have finite multiplicity. As is shown in [24], $\sigma(T^{-1})$ is the spectrum of Problem PL.

Since T is positive and self-adjoint, we know that $\sigma(T) \subset [0, \infty)$ and

$$\begin{aligned} \|T\| &= \max \sigma(T) = \sup \{\langle T(u), u\rangle_A : u \in H_A \text{ with } \|u\|_A = 1\} \\ &= \sup \left\{ \frac{\langle u, u\rangle}{\langle u, u\rangle_A} : u \in H_A \backslash \{0\} \right\}. \end{aligned}$$

This can be expressed directly using the Rayleigh quotient for (1.1),

$$Q_A(u) = \frac{\int_0^1 A(s)u'(s)^2 ds}{\int_0^1 u(s)^2 ds}, \tag{2.8}$$

(we set $Q_A(u) = 0$ if $\int_0^1 u(s)^2 ds = \infty$) and its infimum

$$\Lambda(A) = \inf \{Q_A(u) : u \in H_A \backslash \{0\}\}. \tag{2.9}$$

For $p > 2$ and $\frac{1-p}{2} < \alpha < -\frac{1}{2}$, it is easy to see that for the test functions defined in (2.5),

$$0 < \int_0^1 A(s)u_\alpha'(s)^2 ds < \infty \text{ and } \int_0^1 u_\alpha(s)^2 ds = \infty.$$

Hence

$$\Lambda(A) = 0 \text{ if } p > 2. \tag{2.10}$$

But for $p \in [0, 2]$, it follows from Lemma 2.2 that

$$\Lambda(A) \geq \frac{K_2}{4} > 0. \tag{2.11}$$

Hence,

$$\|T\| = \max \sigma(T) = 1/\Lambda(A). \tag{2.12}$$

and $\Lambda(A)$ is the infimum of the spectrum of Problem PL. The eigenfunctions of Problem PL are precisely the eigenfunctions of the T.

Lemma 2.4 *Let A be a profile with tapering of order $p \in [0,2]$. Then u is an eigenfunction of the Problem PL if and only if $u \in H_A \backslash \{0\}$ and $u = \mu T u$. Furthermore, all eigenvalues of T are simple.*

Theorem 2.5 *Let A be a profile with tapering of order p where $0 \leq p < 2$. Then*

$$\sigma_d(T) = \{\lambda_i : i \in \mathbb{N}\} \text{ and } \sigma_e(T) = \{0\}$$

where $\lambda_{i+1} < \lambda_i, \lambda_1 = \Lambda(A)^{-1}, \lim_{i\to\infty} \lambda_i = 0$ and each λ_i is a simple eigenvalue of T. If φ_i is an eigenfunction of T associated with λ_i then
(a) $\varphi_i \in C^1((0,1]) \cap L^\infty(0,1)$
(b) $\lim_{s\to 0} \varphi_i(s)$ exists. It is finite and non-zero.
(c) φ_i has exactly i zeros in $(0,1]$ and all the zeros of φ_i are simple.

The preceding theorem shows that, for $0 \leq p < 2$, Problem PL behaves like a regular Sturm-Liouville problem, in particular $\sigma_e(T) = \{0\}$. For $p = 2$, the situation is different. Then it is always the case that $\max \sigma_e(T) > 0$ and it may happen that $\sigma_d(T) = \emptyset$. We now give a series of results which justify these statements in a sharper form.

Lemma 2.6 *Let A be a profile with tapering of order 2. Then $\Lambda(A) \leq L/4$ where $L = \lim_{s\to 0} A(s)/s^2$ and, for $\mu > L/4$, all solutions of the equation (1.19) have infinitely many zeros.*

Theorem 2.7 *Let A be a profile with tapering of order 2. Then $\max \sigma_e(T) = \frac{4}{L}$ where $L = \lim_{s\to 0} A(s)/s^2$ and $T : H_A \to H_A$ is not compact.*

Recalling that $\Lambda(A) = \inf \sigma(T^{-1})$ is the infimum of the spectrum of Problem PL, we introduce the notation

$$\Lambda_e(A) = \inf \sigma_e(T^{-1})$$

for the infimum of the essential spectrum of Problem PL.

Theorem 2.8 *Let A be a profile with tapering of order $p = 2$. Then*

$$\frac{K_2}{4} \leq \Lambda(A) \leq \frac{L}{4} = \Lambda_e(A)$$

where $L = \lim_{s\to 0} A(s)/s^2$ and $K_2 = \inf_{0<s\leq 1} A(s)/s^2$. In particular, $\Lambda(A) = \Lambda_e(A) = \frac{L}{4}$ provided that

$$A(s) \geq Ls^2 \text{ for all } s \in (0,1].$$

Remark To show that $\Lambda(A) < \Lambda_e(A)$ it is enough to find one function $u \in H_2$ for which $Q_A(u) < \frac{L}{4}$. As is shown in [24], this can be done provided that

$$\frac{\pi^2 \max_{s\in I} A(s)}{|I|\{2\delta + |I|\}} < L. \tag{2.13}$$

for some interval $I = [\delta, \gamma] \subset (0,1]$.

Finally we show that in some cases there may be no eigenfunctions at all.

Theorem 2.9 *Let A be a profile with tapering of order 2. Suppose that $A \in C^1([0,1])$, with $\lim_{s\to 0} A'(s)/s = 2L$ and that, for all $s \in (0,1]$,*

$$\frac{dh}{ds}(s) \leq 0 \text{ where } h(s) = A(s)\{\int_s^1 A(\tau)^{-1}d\tau\}^2. \tag{2.14}$$

Then the operator $T : H_A \to H_A$ has no eigenvalues and $u \equiv 0$ is the only solution of Problem PL.

Remark 1 For the profile Ls^2, we find that $h(s) = (1-s)^2/L$ and so (2.14) is satisfied and it can also be checked for profiles like $Ls^2 + Cs^q$ where $C > 0$ and $q > 2$.

Remark 2 Let us reformulate the condition (2.14) in terms of the physical variables (1.10) and (1.11) for the problem of a column buckling under its own weight. Suppose that (1.16) holds with $r = q + 3$ so that $p = 2$. Then $A \in C^1([0,1])$, with $\lim_{t\to 0} A'(t)/t = 2L$ provided that $I \in C^1([0,1])$ and that $\lim_{s\to 0} I'(1-s)/s^{r-1}$ exists. We find that $\frac{dh}{dt} \leq 0$ for all $t \in (0,1)$ if and only if

$$\frac{d}{dz}\left\{I(z)[\int_0^z I(\tau)^{-1}d\tau]^2 \int_z^1 S(\tau)d\tau\right\} \geq 0 \text{ for all } z \in (0,1).$$

For the case of a loaded beam, we simply have $A(s) = I(1-s)$ and so (2.14) becomes

$$\frac{d}{dz}\left\{I(z)[\int_0^z I(\tau)^{-1}d\tau]^2\right\} \geq 0 \text{ for all } z \in (0,1).$$

2.3 Energy functional

Let $p \in [0,\infty)$ and, for $u \in H_p$, set

$$\psi(u) = \int_0^1 \{1 - \cos u(s)\}\, ds. \tag{2.15}$$

Clearly

$$0 \leq \psi(u) \leq 2 \text{ for all } u \in H_p. \tag{2.16}$$

Lemma 2.10 *Let A be a profile for a column with tapering of order $p \in [0,\infty)$.*
(i) The functional $\psi : H_A \to \mathbb{R}$ is weakly sequentially continuous and, for $u \in H_A$ and $v \in H_A \cap L^1(0,1)$, the function $g(t) = \psi(u + tv)$ is differentiable on $\mathbb{R}$ with

$$g'(0) = \int_0^1 v(s)\sin u(s)ds.$$

(ii) For $p \in [0,2], \psi \in C^1(H_A)$ with

$$\psi'(u)v = \int_0^1 v(s)\sin u(s)ds \quad \text{for all } u, v \in H_A.$$

Furthermore, $\psi' : H_A \to H_A^*$ is Lipschitz continuous.

Remark For all $p \in [0, \infty)$, Proposition 2.1 (iii) implies that $H_A \cap L^1(0,1)$ is a dense subspace of H_A and it follows from (2.1) and (2.2) that $H_A \subset L^1(0,1)$ for $p < 3$.

Proof (i) Choose $p \in [0, \infty)$ and consider a sequence $\{u_n\}$ such that u_n converges weakly to u in H_p. Then, for any $\varepsilon \in (0,1)$,

$$\begin{aligned}|\psi(u) - \psi(u_n)| &= \left|\int_0^1 \{\cos u(s) - \cos u_n(s)\}\, ds\right| \\ &\leq 2\varepsilon + \int_\varepsilon^1 |\cos u(s) - \cos u_n(s)|\, ds.\end{aligned}$$

Since u_n converges uniformly to u on $[\varepsilon, 1]$, it follows that $\limsup_{n\to\infty} |\psi(u) - \psi(u_n)| \leq 2\varepsilon$ for all $\varepsilon \in (0,1)$. Hence $\psi(u_n) \to \psi(u)$ and so ψ is weakly sequentially continuous.

(ii) For $t \neq 0$ let us consider the quotient

$$\frac{g(t) - g(0)}{t} = \int_0^1 \frac{\cos u(s) - \cos[u(s) + tv(s)]}{t} ds.$$

Note that, for all $s \in [0, 1]$,

$$\left|\frac{\cos u(s) - \cos[u(s) + tv(s)]}{t}\right| \leq |v(s)|$$

and

$$\frac{\cos u(s) - \cos[u(s) + tv(s)]}{t} \to v(s) \sin u(s) \text{ as } t \to 0.$$

For $v \in H_A \cap L^1(0,1)$ it now follows from the Dominated Convergence Theorem that g is differentiable at $t = 0$ with

$$g'(0) = \int_0^1 v(s) \sin u(s) ds.$$

For the differentiability at t_0, it is sufficient to replace u by $u + t_0 v$.

(ii) Suppose that $p \in [0, 2]$. For $u, v \in H_A$,

$$\begin{aligned}\left|\int_0^1 v(s) \sin u(s) ds\right| &\leq \int_0^1 |v(s)|\, |u(s)|\, ds \\ &\leq 4 \|v\|_p \|u\|_p \leq \frac{4}{K_2} \|v\|_A \|u\|_A\end{aligned}$$

by Lemma 2.2 and (2.5). By the Riesz Representation Theorem, for all $u \in H_A$, there is a unique element $G_A(u) \in H_A$ such that

$$\langle G_A(u), v\rangle_A = \int_0^1 v(s) \sin u(s) ds$$

for all $v \in H_A$. Now, there exists a constant K such that

$$|1 - \cos\theta| \leq K\theta^2 \text{ and } |\theta - \sin\theta| \leq K\theta^2 \text{ for all } \theta \in \mathbb{R}.$$

Hence,

$$|\cos u - \cos(u+v) - v\sin u| = |\cos u[1-\cos v] - \sin u[v - \sin v]| \leq 2Kv^2$$

and so

$$\begin{aligned}&\left|\psi(u+v) - \psi(u) - \int_0^1 v(s)\sin u(s)ds\right| \\ &\leq 2K\int_0^1 v(s)^2 ds \leq 8K\|v\|_p^2 \leq \frac{8K}{K_2}\|v\|_A^2\end{aligned}$$

by Lemma 2.2 since $p \leq 2$. This proves that ψ is Fréchet differentiable at u with $\psi'(u)v = \int_0^1 v(s)\sin u(s)ds$ for all $u, v \in H_p$. Furthermore,

$$\begin{aligned}|\psi'(u)v - \psi'(w)v| &\leq \int_0^1 |v(s)|\,|\sin u(s) - \sin w(s)|\,ds \\ &\leq \int_0^1 |v(s)|\,|u(s) - w(s)|\,ds \\ &\leq \left\{\int_0^1 |v(s)|^2\,ds\right\}^{1/2}\left\{\int_0^1 |u(s)-w(s)|^2\,ds\right\}^{1/2} \\ &\leq 4\|v\|_p\|u-w\|_p \leq \frac{4}{K_2}\|v\|_A\|u-w\|_A\,,\end{aligned}$$

for all $u, v, w \in H_p$, showing that $\psi' : H_A \to H_A^*$ is Lipschitz continuous.

Remark 4 In the preceding proof we have shown that, for $p \in [0, 2]$, there is a unique function $G_A = \nabla\psi : H_A \to H_A$ such that

$$\psi'(u)v = \langle G_A(u), v\rangle_A \text{ for all } u, v \in H_A. \tag{2.17}$$

Then $G_A = \nabla\psi : H_A \to H_A$ is Lipschitz continuous and we have the following compactness property.

Lemma 2.11 *If A is a profile for a column with tapering of order $p \in [0, 2]$, then $G_A : H_A \to H_A$ is completely continuous.*

Proof Consider a sequence $\{u_n\}$ such that u_n converges weakly to u in H_A. We must show that $\{G_A(u_n)\}$ converges strongly to $G_A(u)$ in H_A.

For $v \in H_A$,

$$\begin{aligned}&|\langle G_A(u_n) - G_A(u), v\rangle_A| \\ &= \left|\int_0^1 v(s)\{\sin u_n(s) - \sin u(s)\}\,ds\right| \\ &\leq 2\int_0^\varepsilon |v(s)|\,ds + \int_\varepsilon^1 |v(s)|\,|u_n(s) - u(s)|\,ds \\ &\leq |v|_2\left\{2\sqrt{\varepsilon} + \left[\int_\varepsilon^1 |u_n(s) - u(s)|^2\,ds\right]^{1/2}\right\} \\ &\leq \frac{2}{\sqrt{K_2}}\|v\|_A\left\{2\sqrt{\varepsilon} + \left[\int_\varepsilon^1 |u_n(s) - u(s)|^2\,ds\right]^{1/2}\right\}\end{aligned}$$

by Lemma 2.2 and (2.5). Hence,

$$\|G_A(u_n) - G_A(u)\|_A \le \frac{2}{\sqrt{K_2}} \left\{ 2\sqrt{\varepsilon} + \left[\int_\varepsilon^1 |u_n(s) - u(s)|^2 \, ds \right]^{1/2} \right\}$$

and so $\limsup_{n\to\infty} \|G_A(u_n) - G_A(u)\|_A \le 4\sqrt{\frac{\varepsilon}{K_2}}$ for all $\varepsilon \in (0,1)$ since $u_n \to u$ uniformly on $[\varepsilon, 1]$ by Remark 1 following Proposition 2.1. Thus $\|G_A(u_n) - G_A(u)\|_A \to 0$ and we can conclude that G_A is completely continuous.

We now consider the differentiability of G_A for profiles with tapering of order p.

Lemma 2.12 *Let A be a profile with tapering of order $p \in [0,2]$ and consider $u \in H_A$.*
(i) There is a unique bounded linear operator, denoted by $L_A(u) : H_A \to H_A$ such that

$$\langle L_A(u)w, v\rangle_A = \int_0^1 w(s)v(s)\cos u(s)ds \tag{2.18}$$

for all $v, w \in H_A$. Clearly $L_A(u) : H_A \to H_A$ is self-adjoint.
(ii) For $0 \le p < 2$, $L_A(u) : H_A \to H_A$ is the Fréchet derivative of $G_A : H_A \to H_A$ at u and

$$G_A'(u) = L_A(u) : H_A \to H_A \text{ is a compact linear operator.}$$

(iii) For $0 \le p < 2, \psi \in C^2(H_A)$.
(iv) For $p = 2$,

$$\frac{d}{dt} G_A(u + tw) \mid_{t=0} = L_A(u)w$$

for all $u, w \in H_A$. Thus, for all $u \in H_A, L_A(u) : H_A \to H_A$ is the Gâteaux derivative of G_A at u.

Remark 5 Note that $L_A(0)$ coincides with the linear operator T defined by (2.7). It follows from Theorem 2.7 that, when $p = 2$, the bounded linear operator $L_A(0) : H_A \to H_A$ is not compact. From this it follows that for $p = 2$, the mapping $G_A : H_A \to H_A$ cannot be Fréchet differentiable at $u = 0$. Indeed, if it were, part (iv) would imply that $G_A'(0) = L_A(0)$. But then $L_A(0) : H_A \to H_A$ would be compact by Lemma 4.1 of [20] and our Lemma 2.11 which implies that $G_A : H_A \to H_A$ is compact. It follows that for a profile A with tapering of order 2, the functional $\psi \notin C^2(H_A)$.

Proof (i) By Lemma 2.2,

$$\left| \int_0^1 w(s)v(s)\cos u(s)ds \right| \le |w|_2 \, |v|_2 \le \frac{4}{K_2} \|w\|_A \, \|v\|_A$$

for all $u, v, w \in H_p$. The existence and properties of $L_A(u)$ follow from the Riesz Representation Theorem.

(ii) Suppose that $0 \le p < 2$ and let $v, w \in H_A$. Then

$$\begin{aligned}
&\langle G_A(u+w) - G_A(u) - L_A(u)w, v\rangle_A \\
&= \int_0^1 \{\sin[u(s)+w(s)] - \sin u(s) - w(s)\cos u(s)\}\, v(s)ds \\
&= \int_0^1 \{\sin u(s)[\cos w(s) - 1] + \cos u(s)[\sin w(s) - w(s)]\}\, v(s)ds
\end{aligned}$$

and so

$$\begin{aligned}
&|\langle G_A(u+w) - G_A(u) - L_A(u)w, v\rangle_A| \\
&\le \int_0^1 \{|\cos w(s) - 1| + |\sin w(s) - w(s)|\}\, |v(s)|\, ds \\
&\le \left\{\int_0^1 v(s)^2 ds\right\}^{1/2} \left\{\int_0^1 \{|\cos w(s) - 1| + |\sin w(s) - w(s)|\}^2 ds\right\}^{1/2} \\
&\le 2\, \|v\|_p \left\{\int_0^1 \{|\cos w(s) - 1| + |\sin w(s) - w(s)|\}^2 ds\right\}^{1/2}, \qquad (2.19)
\end{aligned}$$

by Lemma 2.2. Since $p < 2$, it follows from (2.1),(2.2) that, given any $\varepsilon > 0$, there exists $t = t(\varepsilon) \in (0,1]$ such that

$$\int_0^t w(s)^2 ds \le \varepsilon\, \|w\|_p^2 \ \text{ for all } w \in H_p$$

from which it follows that

$$\begin{aligned}
&\int_0^t \{|\cos w(s) - 1| + |\sin w(s) - w(s)|\}^2 ds \\
&\le \int_0^t \{|w(s)| + 2\,|w(s)|\}^2 ds \le 9\varepsilon\, \|w\|_p^2 \ \text{ for all } w \in H_p.
\end{aligned}$$

On the other hand, by (2.1) and (2.2), there exists $C(t) > 0$, such that

$$|w(s)| \le C(t)\, \|w\|_p \ \text{ for all } s \in [t,1] \text{ and all } w \in H_p.$$

Therefore, there is a constant K such that

$$\begin{aligned}
&\int_t^1 \{|\cos w(s) - 1| + |\sin w(s) - w(s)|\}^2 ds \\
&\le K \int_t^1 w(s)^4 ds \le K C(t)^4\, \|w\|_p^4 \le \varepsilon\, \|w\|_p^2
\end{aligned}$$

provided that $KC(t)^4\, \|w\|_p^2 \le \varepsilon$. Thus we have that, for any $\varepsilon > 0$, there exists $r(\varepsilon) = \left\{\frac{\varepsilon}{KC(t(\varepsilon))^4}\right\}^{1/2} > 0$ such that

$$\int_0^1 \{|\cos w(s) - 1| + |\sin w(s) - w(s)|\}^2 ds \le 10\varepsilon\, \|w\|_p^2$$

for all $w \in H_p$ satisfying $\|w\|_p \le r(\varepsilon)$.

Using the earlier estimate (2.19), we now have that

$$|\langle G_A(u+w) - G_A(u) - L_A(u)w, v\rangle_A| \leq 2\sqrt{10\varepsilon}\,\|w\|_p\,\|v\|_p$$

for all $u, v, w \in H_p$ with $\|w\|_p \leq r(\varepsilon)$. Recalling that $\|\cdot\|_A$ and $\|\cdot\|_p$ are equivalent norms on H_A, this implies that

$$\frac{\|G_A(u+w) - G_A(u) - L_A(u)w\|_A}{\|w\|_A} \to 0 \text{ as } \|w\|_A \to 0.$$

Thus $L_A(u)$ is indeed the Fréchet derivative of G_A at u.

Since $G_A : H_A \to H_A$ is compact by Lemma 2.11, the compactness of $L_A(u) : H_A \to H_A$ follows from Lemma 4.1 of [20].

(iii) By Lemma 2.10(ii) and part (ii) of the present lemma, we need only show that $\|L_A(u_n) - L_A(u)\| \to 0$ whenever $u_n \to u$ in H_A where

$$\begin{aligned}&\|L_A(u_n) - L_A(u)\| \\ &= \sup\left\{\left|\int_0^1 w(s)^2[\cos u_n(s) - \cos u(s)]ds\right| : \|w\|_A = 1\right\}.\end{aligned}$$

Now as in part (ii), given any $\varepsilon > 0$, there exists $t = t(\varepsilon) \in (0, 1]$ such that

$$\int_0^t w(s)^2 ds \leq \varepsilon\,\|w\|_p^2 \text{ for all } w \in H_p,$$

and hence

$$\begin{aligned}&\int_0^t w(s)^2\,|\cos u_n(s) - \cos u(s)|\,ds \\ &\leq 2\int_0^t w(s)^2 ds \leq 2\varepsilon\,\|w\|_p^2 \text{ for all } w \in H_p.\end{aligned}$$

But if $\|u_n - u\|_p \to 0$, there exists $m = m(t)$ such that $|u_n(s) - u(s)| \leq \varepsilon$ for all $s \in [t, 1]$ and all $n \geq m$. Using Lemma 2.2, it follows that

$$\begin{aligned}\int_t^1 w(s)^2\,|\cos u_n(s) - \cos u(s)|\,ds &\leq \varepsilon\int_t^1 w(s)^2 ds \\ &\leq 2\varepsilon\,\|w\|_p^2\end{aligned}$$

for all $w \in H_p$ provided that $n \geq m$. Combining these two estimates with (2.4), we see that

$$\|L_A(u_n) - L_A(u)\| \leq \frac{4\varepsilon}{K_2}$$

for all $n \geq m(t(\varepsilon))$, establishing the continuity of $L_A : H_A \to B(H_A, H_A)$.

(iv) Let $p=2$ and consider $u,v,w\in H_2$. For $t\neq 0$,

$$
\begin{aligned}
&\left|\left\langle \frac{G_A(u+tw)-G_A(u)}{t} - L_A(u)w, v\right\rangle_A\right| \\
&\leq \int_0^1 \{\left|\frac{\cos tw(s)}{t}-1\right| + \left|\frac{\sin tw(s)}{t} - w(s)\right|\} \left|v(s)\right| ds \\
&\leq \left\{\int_0^1 v(s)^2 ds\right\}^{1/2} \left\{\int_0^1 \left[\left|\frac{\cos tw(s)}{t}-1\right| + \left|\frac{\sin tw(s)}{t} - w(s)\right|\right]^2 ds\right\}^{1/2} \\
&\leq 2\left\|v\right\|_p \left\{\int_0^1 \left[\left|\frac{\cos tw(s)}{t}-1\right| + \left|\frac{\sin tw(s)}{t} - w(s)\right|\right]^2 ds\right\}^{1/2}
\end{aligned}
$$

so

$$
\begin{aligned}
&\left\|\frac{G_A(u+tw)-G_A(u)}{t} - L_A(u)w\right\|_A \\
&\leq \frac{2}{\sqrt{K_2}} \left\{\int_0^1 [\left|\frac{\cos tw(s)-1}{t}\right| + \left|\frac{\sin tw(s)}{t} - w(s)\right|]^2 ds\right\}^{1/2}
\end{aligned}
$$

where

$$
[\left|\frac{\cos tw(s)-1}{t}\right| + \left|\frac{\sin tw(s)}{t} - w(s)\right|]^2 \leq 9w(s)^2
$$

for all $t\neq 0$ and all $s\in(0,1]$. Hence, by dominated convergence,

$$
\left\|\frac{G_A(u+tw)-G_A(u)}{t} - L_A(u)w\right\|_A \to 0 \text{ as } t\to 0
$$

as required

3 Energy minimizing configurations

Throughout this section we consider the Problem P for a profile A with tapering of any order $p\geq 0$ and a constant $\mu>0$. The energy functional (1.17) can be written as

$$
J_\mu(u) = \frac{1}{2}\int_0^1 A(s)u'(s)^2 ds - \mu\psi(u) = \frac{1}{2}\left\|u\right\|_A^2 - \mu\psi(u). \tag{3.1}
$$

where ψ is defined by (2.15). Most of the results of this section are concerned with existence, uniqueness and properties of a configuration u_μ which minimizes J_μ in the space H_A of all admissible configurations. We begin by relating stationary points of J_μ to solutions of Problem P. For $p>2$, J_μ may not be Fréchet differentiable. However for all $p\geq 0$ and $u\in H_A$, we have that

$$
\frac{d}{dt}J_\mu(u+tv)\mid_{t=0} = \int_0^1 A(s)u'(s)v'(s)ds - \mu\int_0^1 v(s)\sin u(s)ds \tag{3.2}
$$

for all $v\in H_A\cap L^1(0,1)$, by Lemma 2.10(i). We recall from Lemma 2.1 that $H_A\cap L^\infty(0,1)$ is dense in H_A. Thus J_μ has directional derivatives at u for all directions in a dense subspace of H_A.

Theorem 3.1 *Let A be a profile with tapering of order $p \geq 0$.*
(i) A function u is a solution of Problem P if and only if $u \in H_A$ and

$$\int_0^1 A(s)u'(s)v'(s)ds = \mu \int_0^1 v(s)\sin u(s)ds \qquad (3.3)$$

for all $v \in H_A \cap L^1(0,1)$.
(ii) For $p \in [0,2]$, $J_\mu \in C^1(H_A)$ and a function u is a solution of Problem P if and only if $u \in H_A$ and $J'_\mu(u) = 0$.

Remark It follows from (2.1) and (2.2), that $H_A \subset L^1(0,1)$ for $p < 3$.
Proof (i) Suppose that u is a solution of Problem P. Then

$$A(s)u'(s) = -\mu \int_0^s \sin u(\tau)d\tau \text{ for } s \in (0,1] \qquad (3.4)$$

and so

$$|A(s)u'(s)| \leq \mu s \text{ for } s \in (0,1]. \qquad (3.5)$$

Also, for $v \in H_A \cap L^1(0,1)$,

$$\begin{aligned}\mu \int_0^1 v(s)\sin u(s)ds &= \mu \lim_{\varepsilon \to 0+} \int_\varepsilon^1 v(s)\sin u(s)ds \\ &= -\lim_{\varepsilon \to 0+} \int_\varepsilon^1 v(s)\left\{A(s)u'(s)\right\}' ds \\ &= \lim_{\varepsilon \to 0+} \{A(\varepsilon)u'(\varepsilon)v(\varepsilon) + \int_\varepsilon^1 A(s)u'(s)v'(s)ds\} \\ &= \int_0^1 A(s)u'(s)v'(s)ds + \lim_{\varepsilon \to 0+} A(\varepsilon)u'(\varepsilon)v(\varepsilon)\end{aligned}$$

Thus $\lim_{s \to 0+} A(s)u'(s)v(s) = l$ exists. If $l \neq 0$, there exists $\delta > 0$ such that

$$|A(s)u'(s)v(s)| \geq |l|/2 \text{ for all } s \in (0,\delta)$$

and hence

$$|l| \leq 2\left|\frac{A(s)u'(s)}{s}\right| |sv(s)| \leq 2\mu |sv(s)|$$

for all $s \in (0,\delta)$, by (3.5). This contradicts the fact that $v \in L^1(0,1)$ and so we must have

$$\lim_{s \to 0+} A(s)u'(s)v(s) = 0 \qquad (3.6)$$

for all $H_A \cap L^1(0,1)$. Thus (??) is satisfied.

Conversely, suppose that $u \in H_A$ and satisfies (??). It follows that $A(s)u'(s)$ admits a generalized derivative on $(0,1)$ and that

$$\{A(s)u'(s)\}' = -\mu \sin u(s) \text{ a.e. on } (0,1).$$

However, since $u \in H_A$, we know that $u \in C((0,1])$ and hence $Au' \in C^1((0,1])$. From the properties of A, this implies that $u \in C^1((0,1])$. Let $v \in C^1([0,1])$ be such that $v(1)=0$ and $v(s)=1$ for all $s \leq 1/2$. Clearly $v \in H_A \cap L^1(0,1)$ and, for any $\varepsilon \in (0,\frac{1}{2})$,

$$\begin{aligned} A(\varepsilon)u'(\varepsilon) &= -\int_\varepsilon^1 A(s)u'(s)v'(s)ds - \int_\varepsilon^1 \{A(s)u'(s)\}' v(s)ds \\ &= -\int_0^1 A(s)u'(s)v'(s)ds + \mu\int_\varepsilon^1 v(s)\sin u(s)ds \end{aligned}$$

since $v' \equiv 01$ on $(0,1/2)$ and $\{A(s)u'(s)\}' = -\mu \sin u(s)$ on $(0,1)$. Using (??), we now have that

$$A(\varepsilon)u'(\varepsilon) = -\mu \int_0^\varepsilon \sin u(s)ds.$$

Hence

$$|A(\varepsilon)u'(\varepsilon)| \leq \mu\varepsilon \text{ for } \varepsilon \in (0,1/2)$$

and, in particular,

$$\lim_{s\to 0} A(s)u'(s) = 0.$$

Thus u is a solution of Problem P.

(ii) By Lemma 2.10(ii), $J_\mu \in C^1(H_A)$ and

$$J'_\mu(u)v = \int_0^1 A(s)u'(s)v'(s)ds - \mu\int_0^1 v(s)\sin u(s)ds$$

for all $v \in H_A$.

Remark We have shown that any solution of Problem P satisfies (3.4), (3.5) and (3.6).

Lemma 3.2 *Let A be a profile with tapering of order $p \geq 0$. Let $u \in H_A$ and, for $s \in (0,1]$, set*

$$\widehat{u}(s) = \begin{cases} \pi \text{ if } u(s) \geq \pi \\ u(s) \text{ if } -\pi < u(s) < \pi \\ -\pi \text{ if } u(s) \leq -\pi \end{cases}. \tag{3.7}$$

Then $\widehat{u} \in H_A$ and $J_\mu(\widehat{u}) < J_\mu(u)$ unless $\widehat{u} \equiv u$.

Proof By Corollary 2.4 in Chapter VI of [11], $\widehat{u}$ admits a generalized derivative on $(0,1)$ and

$$\widehat{u}'(s) = \begin{cases} u'(s) & \text{a.e. where } -\pi < u(s) < \pi \\ 0 & \text{a.e. where } |u(s)| \geq \pi \end{cases}.$$

Hence $\widehat{u} \in H_A$ and

$$J_\mu(\widehat{u}) - J_\mu(u) = -\frac{1}{2}\int_{C(u)} A(s)u'(s)^2 + 2\mu\{1+\cos u(s)\}\, ds$$

where $C(u) = \{s \in (0,1) : |u(s)| > \pi\}$. Clearly $J_\mu(\widehat{u}) - J_\mu(u) < 0$ if $C(u) \neq \emptyset$.

We now come to the first main result concerning configurations of minimum energy.

Theorem 3.3 *Let A be a profile with tapering of order $p \geq 0$ and, for $\mu > 0$, set*

$$m(\mu) = \inf \{J_\mu(u) : u \in H_A\}. \tag{3.8}$$

(i) $m : (0, \infty) \to \mathbb{R}$ is a non-increasing Lipschitz continuous function with $\lim_{\mu \to 0} m(\mu) = 0$.
(ii) There is an element $u_\mu \in H_A$ such that

$$J_\mu(u_\mu) = m(\mu) \text{ and } u_\mu(s) \geq 0 \text{ for all } s \in (0, 1]. \tag{3.9}$$

(iii) u_μ is a solution of Problem P.
(iv) Either $m(\mu) = 0$, in which case $u_\mu \equiv 0$ and $J_\mu(u) > 0$ for all $u \in H_A \backslash \{0\}$; or $m(\mu) < 0$, in which case $0 < u_\mu(s) < \pi$ for all $s \in (0, 1)$, $u'_\mu(s) < 0$ for all $s \in (0, 1]$ and

$$\{u \in H_A : J_\mu(u) = m(\mu)\} = \{\pm u_\mu\}.$$

Thus in both cases, u_μ is uniquely determined by μ for all $\mu > 0$
(v) If $m(\mu) < 0$ and $\lambda > \mu$, then $u_\lambda(s) > u_\mu(s)$ for all $s \in (0.1)$.
(vi) Setting $U(\mu) = u_\mu$, the function $U : (0, \infty) \to H_A$ is continuous.

Remark 1 It is shown below that, for all $p \geq 0$, $\{\mu > 0 : m(\mu) < 0\} \neq \emptyset$ and, in fact,

$$\inf \{\mu > 0 : m(\mu) < 0\} = \Lambda(A) \tag{3.10}$$

where $\Lambda(A)$ is defined by (2.9). See Theorems 3.6, 3.7 and 3.9 below. Recall that $\Lambda(A) = 0$ if and only if $p > 2$. Furthermore, for $\Lambda(A) < \mu < \lambda$, we have

$$J(u_\mu) = m(\mu) < 0 \text{ and so } \psi(u_\mu) > \frac{1}{2\mu} \|u_\mu\|_A^2 > 0,$$

and also,

$$m(\lambda) \leq J_\lambda(u_\mu) = \frac{1}{2} \|u_\mu\|_A^2 - \lambda \psi(u_\mu).$$

It follows that $\lim_{\lambda \to \infty} m(\lambda) = -\infty$. Furthermore, $J_\lambda(u_\mu) < J_\mu(u_\mu) = m(\mu)$ so m is strictly decreasing on $(\Lambda(A), \infty)$.

Remark 2 It follows from Lemma 4.4(i) that $U : (0, \infty) \to L^\infty(0, 1)$ is also a continuous function for profiles with tapering of order $p < 2$. For $p \geq 2$ this is no longer the case since, as is shown in Theorem 3.10,

$$\|u_\mu\|_{L^\infty(0,1)} = \begin{cases} 0 & \text{for } 0 < \mu \leq \Lambda(A) \\ \pi & \text{for } \mu > \Lambda(A) \end{cases}. \tag{3.11}$$

Proof (i) Clearly,

$$J_\mu(u) \geq \frac{1}{2} \|u\|_A^2 - 2\mu \text{ for all } u \in H_A \tag{3.12}$$

and so for all $\mu > 0$,

$$0 = J_\mu(0) \geq m(\mu) \geq -2\mu > -\infty. \tag{3.13}$$

Thus $m(\mu) \to 0$ as $\mu \to 0$. For $0 < \mu < \lambda$ and any $u \in H_A$,

$$J_\mu(u) - J_\lambda(u) = (\lambda - \mu)\psi(u) \geq 0 \text{ so } m(\mu) \geq m(\lambda),$$

whereas

$$(\lambda - \mu)\psi(u) \leq 2(\lambda - \mu) \text{ and so } m(\mu) \leq m(\lambda) + 2(\lambda - \mu).$$

Hence m is non-increasing and Lipschitz continuous on $(0, \infty)$.

(ii) By Lemma 2.10(i), $J_\mu : H_A \to \mathbb{R}$ is weakly sequentially lower semicontinuous. It follows easily from (3.12) that there exists an element $u_\mu \in H_A$ such that $J_\mu(u_\mu) = \min\{J_\mu(u) : u \in H_A\}$. But if $u \in H_A$ then by Corollary 2.4 in Chapter VI of [11], so does $|u|$ and $J_\mu(u) = J_\mu(|u|)$. Consequently we can assume that $u_\mu(s) \geq 0$ for all $s \in (0, 1]$.

(iii) Let $v \in H_A \cap L^\infty(0, 1)$. Then $J_\mu(u_\mu) \leq J_\mu(u_\mu + tv)$ for all $t \in \mathbb{R}$ and it follows from Lemma 2.10(i) that $J_\mu(u_\mu + tv)$ is a differentiable function of t. Hence $\frac{d}{dt} J_\mu(u + tv) \mid_{t=0} = 0$. Using (3.2) and Theorem 3.1, it follows that u_μ is a solution of Problem P.

(iv) Since $J_\mu(0) = 0, m(\mu) \leq 0$. Suppose first that $m(\mu) = 0$ and let $u \in H_A \backslash \{0\}$ be such that $J_\mu(u) = 0$. Then, in the notation of Lemma 3.2, $\widehat{u} \in H_A \backslash \{0\}$ and $0 \leq J_\mu(\widehat{u}) \leq J_\mu(u) = 0$. It follows that from part (iii) of the present result and Theorem 3.1(i) that $\widehat{u}$ satisfies (3.3). But

$$1 - \cos\theta > \frac{1}{2}\theta \sin\theta \text{ for all } \theta \in (0, \pi),$$

from which it follows that

$$J_\mu(\widehat{u}) < \frac{1}{2} \int_0^1 A(s)\widehat{u}'(s)^2 - \mu \widehat{u}(s) \sin \widehat{u}(s) ds$$

and hence that $J_\mu(\widehat{u}) < 0$ since $\widehat{u}$ satisfies (3.3). This contradicts the fact that $J_\mu(\widehat{u}) \geq m(\mu) = 0$ and so we can conclude that $J_\mu(u) > 0$ for all $u \in H_A \backslash \{0\}$ if $m(\mu) = 0$.

Suppose henceforth that $m(\mu) < 0$. By part (iii), u_μ is a solution of Problem P and, by Lemma 3.2, we can suppose that $0 \leq u_\mu(s) \leq \pi$ for all $s \in (0, 1]$. But if $u_\mu(s_0) = 0$ for some $s_0 \in (0, 1)$, it then follows that $u'_\mu(s_0) = 0$ and consequently that $u_\mu \equiv 0$ on $(0, 1)$ by the uniqueness of the Cauchy problem for (1.13). Thus $m(\mu) = J_\mu(u_\mu) = J_\mu(0) = 0$, a contradiction. Hence $u_\mu(s) > 0$ for all $s \in (0, 1)$. Similarly, if $u_\mu(s_0) = \pi$ for some $s_0 \in (0, 1)$, then $u'_\mu(s_0) = 0$ and it follows that $u_\mu \equiv \pi$ on $(0, 1)$ by the uniqueness of the Cauchy problem for (1.13), contradicting the fact that $u_\mu(1) = 0$. Hence $0 < u_\mu(s) < \pi$ for all $s \in (0, 1)$. But then

$$A(s)u'_\mu(s) = -\mu \int_0^s \sin u_\mu(t) dt < 0$$

for all $s \in (0, 1]$, showing that $u'(s) < 0$ in this interval.

Suppose now that $w \in \{u \in H_A : J_\mu(u) = m(\mu)\}$. Then $w \not\equiv 0, w$ is a solution of Problem P and, by Lemma 3.2, $|w(s)| \leq \pi$ for all $s \in (0, 1]$. Furthermore, again by the uniqueness of the Cauchy problem for (1.13), $|w(s)| < \pi$ for all $s \in (0, 1]$ and $w'(1) \neq 0$. But $|w| \in H_A$ and $J_\mu(|w|) = J_\mu(w) = m(\mu)$, so $|w|$

is also solution of Problem P by part (iii). But then $|w| \in C^1((0,1])$ and, if $|w(s)| = 0$ for some $s \in (0,1)$, it follows that $|w|'(s) = 0$ and consequently that $|w| \equiv 0$. Hence we see that w cannot have a zero in $(0,1)$. Suppose first that $w'(1) < 0$. We shall show that in this case $w = u_\mu$.

If $w'(1) = u_\mu'(1)$, we have $w \equiv u_\mu$ by the uniqueness of the solution of the Cauchy problem for (1.13). Let us suppose that $w'(1) < u_\mu'(1)$. Then $w(s) > u_\mu(s)$ in some maximal interval $(z,1)$ and we claim that $z = 0$. Indeed if $z > 0$,we have that $w(z) = u_\mu(z)$ and $w'(z) \geq u_\mu'(z)$. Furthermore, since w and u_μ are both solutions of Problem P, we have that

$$\begin{aligned}
&\mu \int_z^1 \left\{ \frac{\sin u_\mu(s)}{u_\mu(s)} - \frac{\sin w(s)}{w(s)} \right\} u_\mu(s) w(s) ds \\
&= -\int_z^1 \left\{A(s)u_\mu'(s)\right\}' w(s) - \left\{A(s)w'(s)\right\}' u_\mu(s) ds \\
&= A(z)w(z)\left\{u_\mu'(z) - w'(z)\right\}.
\end{aligned}$$

But the function $\frac{\sin\theta}{\theta}$ is strictly decreasing on $[0,\pi]$ and $A(z)w(z) > 0$, so that $u_\mu'(z) > w'(z)$. This contradiction excludes the possibility that $z > 0$ and we can suppose that $w(s) > u_\mu(s)$ for all $s \in (0,1)$. But now we have that

$$\begin{aligned}
0 &< \mu \lim_{a\to 0} \int_a^1 \left\{ \frac{\sin u_\mu(s)}{u_\mu(s)} - \frac{\sin w(s)}{w(s)} \right\} u_\mu(s) w(s) ds \\
&= -\lim_{a\to 0} \int_a^1 \left\{A(s)u_\mu'(s)\right\}' w(s) - \left\{A(s)w'(s)\right\}' u_\mu(s) ds \\
&= \lim_{a\to 0} A(a)u_\mu'(a)w(a) - A(a)w'(a)u_\mu(a) = 0
\end{aligned}$$

by (3.6) and the remark following the proof of Theorem 3.1. Thus the assumption that $w'(1) < u_\mu'(1)$ leads to a contradiction and, in the same way the assumption that $w'(1) > u_\mu'(1)$ also leads to a contradiction. This shows that $w'(1) = u_\mu'(1)$ and so $w = u_\mu$ when $w'(1) < 0$.

If $w'(1) > 0$, we need only apply this conclusion to $-w$ to show that $w = -u_\mu$ in this case.

(v) Let us simplify the notation by setting $u_\mu = v$ and $u_\lambda = u$. Suppose that there is a point $s \in (0,1)$ where $u(s) < v(s)$. Recalling that u and $v \in C((0,1])$, let (a,b) denote a maximal interval in which $u < v$. Then $u(b) = v(b) \geq 0$ and $v'(b) \leq u'(b)$. If $a > 0$ we also have that $u(a) = v(a) \geq 0$ and $v'(a) \geq u'(a)$. For $a > 0$, this means that

$$\int_a^b \{Au'\}' v - \{Av'\}' u ds = A(b)[u'(b) - v'(b)]u(b) - A(a)[u'(a) - v'(a)]u(a) \geq 0$$

whereas by part (iii),

$$\begin{aligned}
\int_a^b \{Au'\}' v - \{Av'\}' u ds &= \int_a^b \left\{ \mu\frac{\sin v(s)}{v(s)} - \lambda\frac{\sin u(s)}{u(s)} \right\} u(s)v(s) ds \\
&\leq (\mu - \lambda) \int_a^b \frac{\sin u(s)}{u(s)} u(s)v(s) ds < 0 \qquad (3.14)
\end{aligned}$$

since $0 < u(s) < v(s) < \pi$ for $s \in (a, b)$ and $\frac{\sin\theta}{\theta}$ is a decreasing function on $(0, \pi)$. From this contradiction we conclude that $a = 0$. But now

$$\begin{aligned}\int_0^b \{Au'\}' v - \{Av'\}' u ds &= A(b)[u'(b) - v'(b)]u(b) - \lim_{\varepsilon\to 0} A(\varepsilon)\{u'(\varepsilon)v(\varepsilon) - u(\varepsilon)v'(\varepsilon)\} \\ &= A(b)[u'(b) - v'(b)]u(b) \geq 0\end{aligned}$$

since u and v are bounded by π and satisfy $\lim_{\varepsilon\to 0} A(\varepsilon)u'(\varepsilon) = \lim_{\varepsilon\to 0} A(\varepsilon)v'(\varepsilon) = 0$. On the other hand (3.14) remains true with $a = 0$ so we again have a contradiction. This shows that $u \geq v$ on $(0, 1]$.

Suppose now that there is a point $s \in (0, 1)$ where $v(s) = u(s)$. Setting $w = u - v$ we have that

$$\begin{aligned}\{A(s)w'(s)\}' &= -\lambda \sin u(s) + \mu \sin v(s) \\ &= (\mu - \lambda) \sin u(s) < 0,\end{aligned}$$

showing that Aw' is strictly decreasing in an open neighbourhood $(s - \delta, s + \delta)$ of s. But $w'(s) = 0$ since $w(s) = 0$ and $w \geq 0$ on $(0, 1)$. Hence we must have $w'(t) > 0$ for $s - \delta < t < s$ and consequently $w(s - \delta) < w(s) = 0$, contradicting the fact that $w \geq 0$ on $(0, 1)$. Hence $u(s) > v(s)$ for all $s \in (0, 1)$.

(vi) Fix $\mu > 0$ and consider a sequence $\{\mu_n\} \subset (0, \infty)$ such that $\mu_n \to \mu$. Set $v_n = u_{\mu_n}$ to simplify the notation. Then by (i),

$$J_{\mu_n}(v_n) = m(\mu_n) \to m(\mu) = J_\mu(u_\mu)$$

Since $\{\psi(v_n)\}$ is a bounded sequence, this implies that $\{||v_n||_A\}$ is also a bounded sequence and so there is a subsequence $\{v_{n_k}\}$ and an element $v \in H_A$ such that $v_{n_k} \rightharpoonup v$ weakly in H_A. From the weak sequential lower-semicontinuity of $J_\mu : H_A \to \mathbb{R}$ it follows that $J_\mu(v) \leq \liminf J_\mu(v_{n_k})$.

But

$$J_\mu(v_{n_k}) = J_{\mu_{n_k}}(v_{n_k}) + (\mu_{n_k} - \mu)\psi(v_{n_k}) \to m(\mu)$$

and so $J_\mu(v) \leq m(\mu)$. Hence $J_\mu(v) = m(\mu)$ and by (iv) this implies that $v = \pm u_\mu$.

On the other hand,

$$\begin{aligned}\frac{1}{2} ||v_{n_k}||_A^2 &= J_{\mu_{n_k}}(v_{n_k}) + \mu_{n_k}\psi(v_{n_k}) \\ &\to m(\mu) + \mu\psi(v) = J_\mu(v) + \mu\psi(v) \\ &= \frac{1}{2} ||v||_A^2\end{aligned}$$

by the weak sequential continuity of $\psi : H_A \to \mathbb{R}$. It follows that $||v_{n_k} - v||_A \to 0$. However $v_{n_k} \geq 0$ on $(0, 1)$ and so $v \geq 0$ on $(0, 1)$, showing that in fact $v = +u_\mu$.

This proves the continuity of $U : (0, \infty) \to H_A$.

Remarks The proof of (iv) shows that 0 and u_μ are the only non-negative solutions of Problem P such that $0 \leq u \leq \pi$ on $(0, 1]$ and we can show that $\pm u_\mu$ form an envelope for all solutions v satisfying $|v| \leq \pi$ on $(0, 1]$. Under some additional assumptions about the form of the profile A it turns out that all solutions are bounded by π.

Corollary 3.4 *Let A be a profile with tapering of order $p \geq 0$. Suppose that $m(\mu) < 0$.*
(i) Let $w \not\equiv 0$ be a solution of Problem P with $0 \leq w \leq \pi$ on $(0,1]$. Then $w \equiv u_\mu$.
(ii) Let $v \not\equiv \pm u_\mu$ be a solution of Problem P with $|v| \leq \pi$ on $(0,1]$. Then $-u_\mu < v < u_\mu$ on $(0,1)$.

Proof (i) This follows from the proof of part (iv) of the preceding theorem. Indeed, having shown that w is a solution of Problem P which satisfies $0 \leq w \leq \pi$ on $(0,1]$, we then deduce from this that $w = u_\mu$.

(ii) This is a straight forward variant of the proof of part (v) of the preceding theorem. Indeed, if we consider a maximal interval (a,b) such that $v > u_\mu$ on (a,b), the same arguments lead to a contradiction. Hence $v \leq u_\mu$ on $(0,1]$. But if there is a point $s \in (0,1)$ such that $v(s) = u_\mu(s)$ this implies that $v'(s) = u_\mu'(s)$ and then the uniqueness of the Cauchy problem for (1.13) yields $v \equiv u_\mu$. Thus $v < u_\mu$ on $(0,1)$ and, replacing v by $-v$, we deduce that $v > -u_\mu$ on $(0,1)$.

Theorem 3.5 *Let A be a profile with tapering of order $p \geq 0$.*
(i) If $u \not\equiv 0$ is a solution of Problem P such that $|u(s)| \leq \pi$ for all $s \in (0,1]$, then $-\pi < u(s) < \pi$ for all $s \in (0,1]$ and $J_\mu(u) < 0$.
(ii) If the profile A has the property that

$$A \text{ is differentiable and } A'(s) \geq 0 \text{ for all } s \in (0,1), \tag{3.15}$$

then every solution $u \not\equiv 0$ of Problem P is such that $-\pi < u(s) < \pi$ for all $s \in (0,1]$. Furthermore, if $0 < s < t < 1$ and $u'(s) = u'(t) = 0$, then $|u(s)| \geq |u(t)|$. If the inequality in (3.15) is strict, then $|u(s)| > |u(t)|$.
(iii) Still supposing that (3.15) is satisfied, let $u \not\equiv 0$ be a solution of Problem P and consider $0 \leq s < t \leq 1$ such that $u(s) = u(t) = 0$. Then $A(s)|u'(s)| \leq A(t)|u'(t)|$ and the inequality is strict unless $A' \equiv 0$ on $[s,t]$.

Remark 1 Let us express the condition (3.15) in terms of the physical variables for the problem of a column buckling under its own weight. Using (1.10) and (1.11) we find that A is differentiable if and only if I is differentiable. Furthermore,

$$A'(t) \geq 0 \text{ for all } t \in (0,1) \iff \frac{d}{dz}\left\{I(z)\int_z^1 S(\tau)d\tau\right\} \leq 0 \text{ for all } z \in (0,1).$$

Note that this is certainly true if I is a non-increasing function of z.

Remark 2 It follows from part (iii) that buckled equilibrium configurations do not cross the vertical axis through the clamped end. Indeed, for the case of a loaded rod discussed in Section 7, the lateral displacement is given by

$$X(s) = \int_s^1 \sin u(\sigma)d\sigma, \text{ see (1.2)},$$

and so, if $t \in (0,1)$ is a stationary point of X we have that $0 = X'(t) = -\sin u(t)$. Since $-\pi < u(t) < \pi$ by part (ii) of the theorem, it follows that $u(t) = 0$. Thus, applying part (iii) to the points t and 1, we find that

$$-A(t)u'(t) \leq A(t)|u'(t)| \leq A(1)|u'(1)|.$$

But, for all $s \in (0, 1]$,

$$A(s)u'(s) = -\mu \int_0^s \sin u(\sigma) d\sigma = -\mu[X(0) - X(s)]$$

and, in particular,

$$\mu[X(0) - X(t)] = -A(t)u'(t) \leq A(1) |u'(1)| .$$

Supposing (without loss of generality) that $u'(1) < 0$, we find that $\mu[X(0) - X(t)] \leq -A(1)u'(1) = \mu X(0)$ since $X(1) = 0$. Hence $X(t) \geq 0$ at every stationary point of X when $u'(1) < 0$ and this implies that $X(s) \geq 0$ for all $s \in [0, 1]$. The inequality is strict for $s \in [0, 1)$ if $A' > 0$ in a neighbourhood if 1. In fact, a similar argument shows that the distance of successive extreme values of $X(s)$ from $X(0)$ increases with s. See Figures 10, 12 and 14.

Proof (i) As in the preceding proofs, the uniqueness of the solution of Cauchy problem for (1.13) implies that $-\pi < u(s) < \pi$ for all $s \in (0, 1]$. Furthermore, by Theorem 3.1(i) we have that

$$\int_0^1 A(s)u'(s)^2 ds = \mu \int_0^1 u(s) \sin u(s) ds.$$

Hence,

$$J_\mu(u) = \mu \int_0^1 \left\{ \frac{1}{2} u(s) \sin u(s) + \cos u(s) - 1 \right\} ds.$$

Since $\frac{1}{2}\theta \sin\theta + \cos\theta - 1 < 0$ for all $\theta \in [-\pi, \pi]\backslash\{0\}$, it follows that $J_\mu(u) < 0$.

(ii) Consider the function

$$V(s) = \frac{1}{2} \left[A(s)u'(s)\right]^2 - \mu A(s)\{1 + \cos u(s)\}. \tag{3.16}$$

For a solution of Problem P, we find that

$$V'(s) = -\mu A'(s)\{1 + \cos u(s)\} \leq 0$$

for all $s \in (0, 1)$ since A is differentiable with $A'(s) \geq 0$ by (3.15). But

$$\lim_{s\to 0} V(s) = -\mu \lim_{s\to 0} A(s)\{1 + \cos u(s)\} \leq 0$$

and so $V(s) \leq 0$ for all $s \in (0, 1]$. Thus,

$$\frac{1}{2} \left[A(s)u'(s)\right]^2 \leq \mu A(s)[1 + \cos u(s)],$$

showing that $u'(z) = 0$ whenever $u(z) = \pm\pi$. By the uniqueness of the solution of the Cauchy problem for (1.13), it follows that $u(s) = u(z)$ for all $s \in (0, 1]$ if there is a point $z \in (0, 1)$ such that $u(z) = \pm\pi$. This is impossible since $u(1) = 0$ and so we conclude that $|u(s)| < \pi$ for all $s \in (0, 1]$.

Now consider the function

$$W(s) = \frac{1}{2A(s)} \left[A(s)u'(s)\right]^2 + \mu[1 - \cos u(s)] \tag{3.17}$$

which is differentiable on $(0,1)$ with

$$W'(s) = -\frac{1}{2}A'(s)u'(s)^2 \le 0 \tag{3.18}$$

for a solution u of Problem P. Hence if $0 < s < t < 1$ and $u'(s) = u'(t) = 0$,

$$\mu[1 - \cos u(s)] = W(s) \ge W(t) = \mu[1 - \cos u(t)]$$

which implies that $|u(s)| \ge |u(t)|$ since $u(s)$ and $u(t) \in (-\pi, \pi)$. If the inequality in (3.15) is strict, then $W' < 0$ on $(0,1)$, except at the zeros of u', and so $|u(s)| > |u(t)|$.

(iii) Finally we consider the function

$$Z(s) = \frac{1}{2}[A(s)u'(s)]^2 - \mu A(s)[-1 + \cos u(s)].$$

For a solution u of Problem P,

$$Z'(s) = -\mu A'(s)[-1 + \cos u(s)] \ge 0 \text{ for all } s \in (0,1)$$

by (3.15). The conclusion follows easily from this.

According to Theorem 3.3, the energy minimizing configuration is a buckled state precisely when $m(\mu) < 0$. The next few results determine when this occurs.

Theorem 3.6 *If A is a profile with tapering of order $p > 2$, $-2\mu \le m(\mu) < 0$ for all $\mu > 0$.*

Proof Recall that the function u_α defined in (2.5) belongs to H_p provided $\alpha > \frac{1-p}{2}$. Furthermore, the function $u_\alpha(s)^2$ is integrable on $(0,1)$ provided that $\alpha > -\frac{1}{2}$ and we observe that $-\frac{1}{2} > \frac{1-p}{2}$ since $p > 2$. Choosing $\alpha > -\frac{1}{2}$ and $t > 0$,

$$\frac{J_\mu(tu_\alpha)}{t^2} = \frac{1}{2}\int_0^1 A(s)u_\alpha'(s)^2 ds - \mu \int_0^1 \frac{1 - \cos[tu_\alpha(s)]}{t^2} ds$$

where

$$\left| \frac{1 - \cos[tu_\alpha(s)]}{t^2} \right| \le \frac{u_\alpha(s)^2}{2}, \tag{3.19}$$

since $0 \le 1 - \cos\theta \le \theta^2/2$ for all $\theta \in \mathbb{R}$. Using the Dominated Convergence Theorem, it follows that

$$\begin{aligned} \lim_{t\to 0+} \frac{J_\mu(tu_\alpha)}{t^2} &= \frac{1}{2}\left\{\int_0^1 A(s)u_\alpha'(s)^2 ds - \mu \int_0^1 u_\alpha(s)^2 ds\right\} \\ &\le \frac{1}{2}\left\{K_1 \int_0^1 s^p u_\alpha'(s)^2 ds - \mu \int_0^1 u_\alpha(s)^2 ds\right\}. \end{aligned}$$

As $\alpha \to -\frac{1}{2}$ from above $\int_0^1 u_\alpha(s)^2 ds \to +\infty$ whereas $\int_0^1 s^p u_\alpha'(s)^2 ds$ remains bounded since $-\frac{1}{2} > \frac{1-p}{2}$. It follows that for any $\mu > 0$, we can choose $\alpha > -\frac{1}{2}$ such that $\lim_{t\to 0+} \frac{J_\mu(tu_\alpha)}{t^2} < 0$, and this shows that we can choose $t > 0$ such that $J_\mu(tu_\alpha) < 0$. Since $tu_\alpha \in H_p = H_A$, this means that $m(\mu) < 0$.

Remark For $p \in [0,2], m(\mu) = 0$ for small μ and $m(\mu) < 0$ for large μ. As we now show the change occurs at the critical value $\mu = \Lambda(A)$ where $\Lambda(A)$ is the infimum of the Rayleigh quotient $Q_A(u)$ discussed in Section 3. Recall from (2.11) and (2.10) that $\Lambda(A) > 0$ for $p \in [0,2]$ and $\Lambda(A) = 0$ for $p > 2$. The main conclusions are given in Theorem 3.9, but we begin with some preparatory results.

Theorem 3.7 *Let A be a profile with tapering of order $p \in [0,2]$. For $\mu \leq \Lambda(A)$, the Problem P has only the trivial solution $u \equiv 0$ and $J_\mu(u) > 0$ for all $u \in H_A \backslash \{0\}$.*

Proof Let u be a non-trivial solution of Problem P for some $\mu \leq \Lambda(A)$. By (3.3),

$$\int_0^1 A(s)u'(s)^2 ds = \mu \int_0^1 u(s)\sin u(s) ds$$
$$< \mu \int_0^1 u(s)^2 ds \leq \frac{\mu}{\Lambda(A)} \int_0^1 A(s)u'(s)^2 ds$$

since $\theta \sin\theta < \theta^2$ for all $\theta \neq 0$. Hence $\mu > \Lambda(A)$.

Thus, for $\mu \leq \Lambda(A)$, we must have $u_\mu \equiv 0$ and by Theorem 3.3(iv), this means that $m(\mu) = 0$ and $J_\mu(u) > 0$ for any $u \in H_A \backslash \{0\}$.

Lemma 3.8 *Let A be a profile with tapering of order $p \in [0,2]$ and let u be a solution of Problem P.*
(i)

$$\int_0^1 u(s)^2 ds \leq \left[\frac{\mu}{\Lambda(A)}\right]^2.$$

(ii) If $\mu > \Lambda(A)$,

$$J_\mu(u) \geq \frac{[\Lambda(A) - \mu]}{2}\left[\frac{\mu}{\Lambda(A)}\right]^2.$$

In particular,

$$\frac{[\Lambda(A) - \mu]}{2}\left[\frac{\mu}{\Lambda(A)}\right]^2 \leq m(\mu) \leq 0.$$

Proof We may as well assume that $u \not\equiv 0$.
(i) By the definition of $\Lambda(A)$,

$$\Lambda(A) \int_0^1 u(s)^2 ds \leq \int_0^1 A(s)u'(s)^2 ds$$
$$= \mu \int_0^1 u(s)\sin u(s) ds, \text{ by (3.3)}$$
$$\leq \mu \int_0^1 |u(s)|\, ds \leq \mu \left\{\int_0^1 u(s)^2\right\}^{1/2}$$

and so

$$\left\{\int_0^1 u(s)^2 ds\right\}^{1/2} \leq \frac{\mu}{\Lambda(A)}.$$

(ii) For $u \in H_A$,

$$\begin{aligned} J_\mu(u) &\geq \frac{\Lambda(A)}{2}\int_0^1 u(s)^2 ds - \mu\int_0^1 [1-\cos u(s)]\, ds \\ &\geq \frac{\Lambda(A)}{2}\int_0^1 u(s)^2 ds - \frac{\mu}{2}\int_0^1 u(s)^2 ds \\ &= \frac{[\Lambda(A)-\mu]}{2}\int_0^1 u(s)^2 ds \end{aligned}$$

since $\cos\theta \geq 1 - \frac{1}{2}\theta^2$ for all $\theta \in \mathbb{R}$. Hence,

$$J_\mu(u) \geq \frac{[\Lambda(A)-\mu]}{2}\left[\frac{\mu}{\Lambda(A)}\right]^2$$

by part (i) if $\Lambda(A) - \mu < 0$.

Theorem 3.9 *(bifurcation to the right at $\Lambda(A)$) Let A be a profile with tapering of order $p \geq 0$.*
(i) If $\mu > \Lambda(A)$ then $J_\mu(u_\mu) = m(\mu) < 0$, and $\lim_{\mu\to\Lambda(A)+} m(\mu) = 0$. For $\mu \leq \Lambda(A)$, $m(\mu) = 0$ and the only solution of Problem P is $u \equiv 0$.
(ii) If $\{v_n\}$ is a solution of Problem P for $\mu_n > \Lambda(A)$ where $J_{\mu_n}(v_n) \leq 0$ and $\lim_{n\to\infty}\mu_n = \Lambda(A)$, then $\lim_{n\to\infty} J_{\mu_n}(v_n) = \lim_{n\to\infty}\|v_n\|_A = 0$ and $v_n \to 0$ uniformly on compact subsets $(0,1]$. In particular,

$$\lim_{\mu\to\Lambda(A)+} J_\mu(u_\mu) = \lim_{\mu\to\Lambda(A)+}\|u_\mu\|_A = 0. \tag{3.20}$$

(iii) $\lim_{\mu\to\infty} m(\mu) = \lim_{\mu\to\infty} J_\mu(u_\mu) = -\infty$ and, for any $t \in (0,1)$, $u_\mu(s) \to \pi$ as $\mu \to \infty$, uniformly on $(0,t]$.

Remark Using Theorem 3.3(vi), we see that the energy minimizing solutions u_μ form a continuous curve in H_A which bifurcates from the solution $u \equiv 0$ at $\mu = \Lambda(A)$.

Proof (i) For $p > 2$, $\Lambda(A) = 0$ and the result follows from Theorems 3.3(i) and 3.6.

Consider the case $p \in [0,2]$. By Lemma 3.8 we have

$$0 \geq m(\mu) \geq \frac{[\Lambda(A)-\mu]}{2}\left[\frac{\mu}{\Lambda(A)}\right]^2$$

and so $\lim_{\mu\to\Lambda(A)+} m(\mu) = 0$.

Choose $\varepsilon > 0$ such that $\Lambda(A)+\varepsilon < \mu$. By the definition of $\Lambda(A)$, there exists an element $u \in H_A \backslash \{0\}$ such that

$$\int_0^1 A(s)u'(s)^2 ds < \{\Lambda(A)+\varepsilon\}\int_0^1 u(s)^2 ds. \tag{3.21}$$

Then, as in (3.19), for $t > 0$ we have that

$$\left| \frac{1 - \cos[tu(s)]}{t^2} \right| \leq \frac{u(s)^2}{2}$$

where, by (2.6), the function on the right hand side of the inequality is integrable on $(0, 1)$. Using the Dominated Convergence Theorem, it follows that

$$\begin{aligned} \lim_{t \to 0+} \frac{J_\mu(tu)}{t^2} &= \frac{1}{2} \int_0^1 A(s) u'(s)^2 ds - \frac{\mu}{2} \int_0^1 u(s)^2 ds \\ &< \frac{1}{2} \{\Lambda(A) + \varepsilon - \mu\} \int_0^1 u(s)^2 ds < 0. \end{aligned}$$

Hence $m(\mu) < 0$.

For the remaining assertions in this part we need only appeal to Theorem 3.7.

(ii) Since $0 \geq J_{\mu_n}(v_n) \geq m(\mu_n)$, it follows from part (i) that $\lim_{n\to\infty} J_{\mu_n}(v_n) = 0$.

For $p > 2$, we have that $\lim_{n\to\infty} \mu_n = \Lambda(A) = 0$ and

$$\begin{aligned} \frac{1}{2} \|v_n\|_A^2 &= J_{\mu_n}(v_n) + \mu_n \int_0^1 [1 - \cos v_n(s)]\, ds \\ &\leq 2\mu_n, \end{aligned}$$

so $\lim_{n\to\infty} \|v_n\|_A = 0$.

Now consider the case where $0 \leq p \leq 2$. By (3.3) and Lemma 3.8(i),

$$\begin{aligned} \int_0^1 A(s) v_n'(s)^2 ds &= \mu_n \int_0^1 v_n(s) \sin v_n(s) ds \\ &\leq \mu_n \int_0^1 v_n(s)^2 ds \leq \mu_n \left[\frac{\mu_n}{\Lambda(A)}\right]^2, \end{aligned}$$

showing that $\{v_n\}$ is a bounded sequence in H_A. Passing to a subsequence, we can suppose that $v_{n_k} \rightharpoonup v$ weakly in H_A for some element $v \in H_A$. But

$$\begin{aligned} 0 &\geq \liminf_{n_k\to\infty} J_{\mu_{n_k}}(v_{n_k}) = \liminf_{n_k\to\infty} \left\{ \frac{1}{2} \|v_{n_k}\|_A^2 - \mu_{n_k} \psi(v_{n_k}) \right\} \\ &\geq \frac{1}{2} \|v\|_A^2 - \Lambda(A)\psi(v) = J_{\Lambda(A)}(v) \end{aligned}$$

by Lemma 2.10(i). Referring to part (i) we see that $m(\Lambda(A)) = 0$ and hence that $v = 0$ since $J_{\Lambda(A)}(v) = m(\Lambda(A))$. But Lemma 2.10(i) now implies that $\psi(v_{n_k}) \to 0$ and so

$$\frac{1}{2} \|v_{n_k}\|_A^2 = J_{\mu_{n_k}}(v_{n_k}) + \mu_{n_k} \psi(v_{n_k}) \to 0,$$

since we have already shown that $J_{\mu_n}(v_n) \to 0$. In fact, this argument proves that every subsequence of $\{v_n\}$ contains a subsequence converging strongly to 0 in H_A. Thus $\lim_{n\to\infty} \|v_n\|_A = 0$.

The remaining conclusions in this part follow from what has already been proved.

(iii) We have shown in Remark 1 following Theorem 3.3 that $\lim_{\mu\to\infty} m(\mu) = -\infty$.

By Theorem 3.3(iv) and (v), $\lim_{\mu\to\infty} u_\mu(s) = l(s)$ where $0 < l(s) \leq \pi$ for all $s \in (0,1)$ and l is a non-increasing function of s. Integrating (3.4), we find that, for $z > 0$,

$$\frac{1}{\mu} u_\mu(z) = \int_z^1 A(s)^{-1}\{\int_0^s \sin u_\mu(t)dt\}ds.$$

Letting $\mu \to \infty$, the Dominated Convergence Theorem yields

$$\int_z^1 A(s)^{-1}\{\int_0^s \sin l(t)dt\}ds = 0$$

for all $z > 0$, where $\sin l(t) \geq 0$ and $A(t) > 0$ for $t \in (0,1]$. It follows that $\sin l(t) = 0$ for all $t \in (0,1)$ and hence that $l(t) = \pi$ for $t \in (0,1)$. In particular, given any $t \in (0,1)$ and any $\varepsilon > 0$, there exists $\lambda > \Lambda(A)$ such that $u_\lambda(t) \geq \pi - \varepsilon$. Hence, by the monotonicity properties established in parts (iv) and (v) of Theorem 3.3,

$$\pi > u_\mu(s) \geq u_\lambda(s) \geq u_\lambda(t) \geq \pi - \varepsilon$$

for all $s \in (0,t]$ and all $\mu \geq \lambda$, showing that $u_\mu(s) \to \pi$ as $\mu \to \infty$, uniformly for $s \in (0,t]$.

Remark The above result shows that for tapering of any order $p \geq 0$, there is bifurcation of a buckled configuration with minimum energy at $\mu = \Lambda(A)$. Indeed, for the case of a loaded rod discussed in Section 7, the parametric representation of this configuration is given by (1.2) and

$$\begin{aligned}&\max_{0\leq s\leq 1} \|r_\mu(s) - (0, 1-s)\| \\ &\leq \{\int_0^1 \sin^2 u_\mu(t) + [1-\cos^2 u_\mu(t)]dt\}^{1/2} \to 0 \text{ as } \mu \to \Lambda(A)+\end{aligned}$$

by the Dominated Convergence Theorem since the fact that $\|u_\mu\|_A \to 0$ as $\mu \to \Lambda(A)+$ implies that $u_\mu \to 0$ pointwise on $(0,1)$. Furthermore the maximum lateral deflection X_μ occurs at the free end and

$$X_\mu = \int_0^1 \sin u_\mu(t)dt \to 0 \text{ as } \mu \to \Lambda(A)+.$$

The maximum height of the buckled column Y_μ occurs at the point $r_\mu(s_\mu)$ where s_μ is the unique value where $u_\mu(s_\mu) = \frac{\pi}{2}$ if $\lim_{s\to 0} u_\mu(s) > \frac{\pi}{2}$ and $s_\mu = 0$ if $\lim_{s\to 0} u_\mu(s) \leq \frac{\pi}{2}$. Hence, in both cases,

$$s_\mu \to 0 \text{ and } Y_\mu = \int_{s_\mu}^1 \cos u_\mu(t)dt \to 1 \text{ as } \mu \to \Lambda(A)+.$$

However, as we now show, there is a dramatic difference between the form of the buckled states for $p \geq 2$ compared to those for $p < 2$. In the former case the free end of the column always points vertically downwards, whereas for $p < 2$, it becomes closer to the vertically upright position with its free end being its highest point as $\mu \to \Lambda(A)+$.

Theorem 3.10 *Let A be a profile with tapering of order $p \geq 0$.*
(i) If $p \geq 2$, then

$$\lim_{s \to 0} u_\mu(s) = \pi \text{ for all } \mu > \Lambda(A).$$

(ii) If $0 \leq p < 2$, then

$$\lim_{s \to 0} u_\mu(s) < \pi \text{ for all } \mu > \Lambda(A).$$

(iii) For $0 \leq p < \infty$ and any bounded solution u of Problem P, we have that

$$\sin \|u\|_{L^\infty(0,1)} \geq 0.$$

Proof Set $u = u_\mu$ and recall from Theorem 3.3 that $u(s) < \pi$ for all $s > 0$ and that $\lim_{s \to 0} u(s) = l$ exists with $0 < l \leq \pi$.

(i) Suppose that $p \geq 2$. If $l < \pi$, there exists $z > 0$ such that $\sin u(s) \geq \delta > 0$ for all $s \in (0, z]$ and hence

$$\{A(s)u'(s)\}' \leq -\mu\delta \text{ for all } s \in (0, z].$$

Thus

$$A(s)u'(s) \leq -\mu\delta s$$

and so

$$u'(s) \leq -\frac{\mu\delta s}{A(s)} \leq -\frac{\mu\delta s^{1-p}}{K_1}$$

for all $s \in (0, z]$. For $p \geq 2$, this implies that

$$l - u(z) = -\int_0^z u'(s)ds = \infty,$$

in contradiction with the fact that $l < \pi$. Hence $l = \pi$ when $p \geq 2$.

(ii) Suppose now that $0 \leq p < 2$ and set $v(s) = \pi - u(s)$. Then $v(s) \geq 0$, $\sin v(s) = \sin u(s)$ and

$$\begin{aligned} A(s)v'(s) &= \mu \int_0^s \sin v(t)dt \\ &\leq \mu s \max_{0<t\leq s} v(t) \end{aligned}$$

so that

$$v'(s) \leq \mu Q(s) \max_{0<t\leq s} v(t)$$

where

$$Q(s) = sA(s)^{-1} \leq \frac{s^{1-p}}{K_2}. \tag{3.22}$$

Hence, if $l = \pi, \lim_{s \to 0} v(s) = 0$ and

$$0 \leq v(s) \leq \mu \max_{0 < t \leq s} v(t) \int_0^s Q(\tau) d\tau.$$

Since $Q \in L^1(0,1)$, it follows easily from this that there exists $s > 0$ such that $\max_{0<t\leq s} |v(t)| = 0$. But then $u(s) = \pi$, which is again a contradiction.

(iii) Suppose that there exists $s \in (0,1)$ such that $u(s) = \pm \|u\|_{L^\infty(0,1)}$ and that $\sin \|u\|_{L^\infty(0,1)} < 0$. Replacing u by $-u$ if necessary, we can assume that $u(s) = \|u\|_{L^\infty(0,1)}$. Then $\{A(s)u'(s)\}' > 0$ and so there is a $\delta > 0$ such that Au' is strictly increasing on $(s-\delta, s+\delta)$. But $u'(s) = 0$ and so $Au' < 0$ on $(s-\delta, s)$. This means that u is strictly decreasing on $(s-\delta, s)$ contradicting the fact that $u(s) = \|u\|_{L^\infty(0,1)}$.

Suppose now that $|u(s)| < \|u\|_{L^\infty(0,1)}$ for all $s > 0$ and that $\sin \|u\|_{L^\infty(0,1)} < 0$. In this case we can assume that $\|u\|_{L^\infty(0,1)} = \limsup_{s \to 0} u(s)$.
If $\liminf_{s \to 0} u(s) < \|u\|_{L^\infty(0,1)}$, there is a sequence $\{s_n\}$ of local maxima of u such that $s_n \to 0$ and $\lim u(s_n) = \|u\|_{L^\infty(0,1)}$. For large enough n, $\sin u(s_n) < 0$ and this leads to a contradiction as above. Hence $\|u\|_{L^\infty(0,1)} = \lim_{s \to 0} u(s)$ and now there exists $\delta > 0$ such that $\sin u(s) < 0$ for all $s \in (0,\delta)$. This means that $\{Au'\}' > 0$ on $(0,\delta)$ which implies that $Au' > 0$ on $(0,\delta)$ since $\lim_{s\to 0} A(s)u'(s) = 0$. Thus u is strictly increasing on $(0,\delta)$, contradicting the fact that $\|u\|_{L^\infty(0,1)} = \lim_{s\to 0} u(s)$. This completes the proof.

For $p < 2$, $\lim_{s \to 0} u(s)$ can be arbitrarily close to 0. As a first step towards proving this we establish a result on the regularity of solutions of Problem P.

Lemma 3.11 *Let A be a profile with tapering of order $p < 3$. Consider $q \geq 0$ such that $q \in (2p-3, p]$ and set $r(q) = \max\left\{0, p + \frac{q-3}{2}\right\}$. For any $r \in (r(q), q]$ there is a constant $D_A(q,r)$ such that, if (μ, u) is a solution of Problem P and $u \in H_q$, then $u \in H_r$ and*

$$\|u\|_r \leq D_A(q,r) \left\{ \mu \|u\|_q \right\}^{1/2}. \tag{3.23}$$

Proof Multiplying (1.13) by $A(s)u'(s)$ and integrating, we find that

$$\begin{aligned}
\frac{1}{2}\{A(s)u'(s)\}^2 &= -\mu \int_0^s A(t)u'(t)\sin u(t)dt \\
&\leq \mu K_1 \int_0^s t^p |u'(t)|\, dt \\
&\leq \mu K_1 \left\{ \int_0^s t^q u'(t)^2 dt \right\}^{1/2} \left\{ \int_0^s t^{2p-q} dt \right\}^{1/2} \\
&= \frac{\mu K_1}{\sqrt{2p-q+1}} s^{p+\frac{1-q}{2}} \|u\|_q
\end{aligned}$$

and so

$$s^r u'(s)^2 \leq \frac{2\mu K_1}{K_2^2 \sqrt{2p-q+1}} s^{p+\frac{1-q}{2}+r-2p} \|u\|_q.$$

Noting that $\frac{1-q}{2} + r - p > -1$ for $r > r(q)$ we see that

$$\|u\|_r^2 \leq \frac{2K_1}{K_2^2(\frac{3-q}{2} + r - p)\sqrt{2p-q+1}} \mu \|u\|_q$$

as required.

This result has several useful consequences. First of all we note that it gives an "a priori" bound for all solutions of Problem P.

Corollary 3.12 *Let A be a profile with tapering of order $p < 3$. Then there is a constant $D(p)$ such that*

$$||u||_A \leq \mu D(p) \tag{3.24}$$

for all solutions (μ, u) of Problem P.

Proof Since $p < 3$, we can set $v = u$ in (3.3). For $0 \leq p \leq 2$, it follows easily from Lemma 2.2 and (2.5)that

$$||u||_A \leq \frac{2\mu}{\sqrt{K_2}}.$$

For $2 < p < 3$, it suffices to put $r = q = p$ in the estimate (3.23).

Remark Observe that the condition $p < 3$ is always satisfied when Problem P is used to treat the buckling of a rod with geometrically similar cross-sections under its own weight. See Remark 3 following the statement of Problem P in the Introduction, and note that $\frac{3q+1}{q+2} < 3$ for all $q \geq 0$.

Theorem 3.13 *Let A be a profile with tapering of order $p \in [0, 2)$. Then all solutions of Problem P are bounded and there are constants $D > 0$ and $\alpha, \beta \in (0, 1]$ such that*

$$||u||_{L^\infty(0,1)} \leq D\mu^\alpha \, ||u||_p^\beta \tag{3.25}$$

for all solutions (μ, u) of Problem P. In particular,

$$||u_\mu||_{L^\infty(0,1)} \to 0 \text{ as } \mu \to \Lambda(A)+ . \tag{3.26}$$

Proof For $0 \leq p < 1$, the result follows immediately from (2.1) and Corollary 3.12.

For $1 \leq p < 5/3$, we use Lemma 3.11 with $q = p$. Then $r(p) = \frac{3}{2}(p-1) < 1$ and we have that $u \in H_r$ and $||u||_r \leq D_A(p, r)\left\{\mu \, ||u||_p\right\}^{1/2}$ for all $r \in (r(p), p]$. Since H_r is continuously embedded in $L^\infty(0, 1)$ for $r < 1$, the result follows.

For $5/3 \leq p < 2$, we observe that $2p - 3 > 0$. Setting $r_0 = p$, we define a sequence $\{r_i\}$ as follows. If $r_i > 2p - 3$, we set $r_{i+1} = p + \frac{r_i - 3}{2}$. Noting that $r_{i+1} > 0$, we see that $r_{i+1} = r(r_i)$ in the notation of Lemma 3.11. Furthermore $r_{i+1} < r_i$. Hence there are two cases which can occur. Either there is a first integer j such that $r_j \leq 2p - 3$, or else $r_i > 2p - 3$ for all $i \in \mathbb{N}$. In the first case $r_j < 1$ since $p < 2$ and, using Lemma 3.11 recursively we obtain the desired conclusion. Otherwise $\{r_i : i \in \mathbb{N}\}$ is a decreasing sequence and we set $R = \lim_{i\to\infty} r_i$. Then $R = p + \frac{R-3}{2}$, showing that $R = 2p - 3 < 1$. Hence, also in this case there is an integer j such that $0 < 2p - 3 < r_j < 1$ and so we can conclude as in the first case.

Recalling from Theorem 3.9 that $\lim_{\mu\to\Lambda(A)+} ||u_\mu||_p = 0$, we see that $||u_\mu||_{L^\infty(0,1)} \to 0$ as $\mu \to \Lambda(A)+$.

4 Bifurcation for sub-critical tapering $(0 \leq p < 2)$

In the preceding section we have seen that, for any profile A with tapering of order $p \geq 0$, a branch of energy minimizing positive solutions $\{u_\mu : \mu > \Lambda\}$ of Problem P bifurcates to the right from the trivial solution $u \equiv 0$ at $\mu = \Lambda(A)$. In this section we discuss all bifurcations from the trivial solution in the case $p < 2$. We use the notation of Theorem 2.5 for the eigenvalues of the operator T defined by (2.7). For $p < 2$, the singularity at $s = 0$ is sufficiently weak so that we obtain global branches emanating from all of the eigenvalues $\left\{\mu_i = \frac{1}{\lambda_i} : i \in \mathbb{N}\right\}$ of Problem PL given by Theorem 2.5. We recall that an eigenfunction φ_i associated with μ_i has exactly i zeros in $(0, 1]$. The solutions on the branch bifurcating from μ_i also have exactly i zeros, just as in the regular case treated originally by Crandall and Rabinowitz [9], and [22]. We begin with a local result and then turn to the global behaviour.

Given a profile A with tapering of order $p \geq 0$, let

$$E = \{(\mu, u) \in \mathbb{R} \times H_A : u \neq 0 \text{ and } (\mu, u) \text{ is a solution of Problem P}\}. \tag{4.1}$$

Recall that (μ, u) is a solution of Problem P if and only if $F(\mu, u) = 0$ where $F : \mathbb{R} \times H_A \to H_A$ is defined by

$$F(\mu, u) = u - \mu G_A(u) \tag{4.2}$$

and G_A is defined by (2.17). For $0 \leq p < 2$, it follows from Lemma 2.12 that $F \in C^1(\mathbb{R} \times H_A, H_A)$ with $D_u F(\mu, 0) = I - \mu T$ where $T \in B(H_A, H_A)$ is defined by (2.7). Furthermore, G_A and $T = L_A(0)$ are compact mappings from H_A into itself. It follows that closed bounded subsets of $E \cup [\mathbb{R} \times \{0\}]$ are compact subsets of $\mathbb{R} \times H_A$.

Before proceeding let us state a form of the Sturm comparison theorem which is appropriate for our context and which will be used repeatedly in what follows.

Proposition 4.1 *Let A be a profile with tapering of order $p \in [0, 2)$ and let $q, Q \in C((0, 1]) \cap L^\infty(0, 1)$ with $q < Q$ on $(0, 1)$, except on a set of measure zero. Let $u, v \in C^1((0, 1]) \cap L^\infty(0, 1) \setminus \{0\}$ be such that $Au', Av' \in C^1((0, 1])$ and*

$$\{A(s)u'(s)\}' + q(s)u(s) = 0 \text{ with } \lim_{s \to 0} A(s)u'(s) = u(1) = 0 \tag{4.3}$$

and

$$\{A(s)v'(s)\}' + Q(s)v(s) = 0 \text{ with } \lim_{s \to 0} A(s)v'(s) = v(1) = 0. \tag{4.4}$$

Then u and v have only a finite number of zeros in $(0, 1]$ and v has more zeros than u in $(0, 1]$.

Proof Let $0 < s_1 < s_2 \leq 1$ be successive zeros of u and suppose that v has no zero in (s_1, s_2). We can assume that $u, v > 0$ on (s_1, s_2) with $u'(s_2) < 0 < u'(s_1)$. Then

$$\begin{aligned} 0 &\geq A(s_2)u'(s_2)v(s_2) - A(s_1)u'(s_1)v(s_1) \\ &= \int_{s_1}^{s_2} \{Q(s) - q(s)\}u(s)v(s)ds > 0. \end{aligned}$$

Hence v has at least one zero in (s_1, s_2). Now by Theorem 2.5, there exists an eigenvalue $\mu_i = 1/\lambda_i$ of Problem PL such that $Q(s) < \mu_i$ for all $s \in (0,1]$. But the corresponding eigenfunction φ_i has exactly i zeros in $(0,1]$ and so we can conclude that u and v have at most i zeros in $(0,1]$. Let n be the number of zeros of u in $(0,1]$ and let τ be the smallest zero of u in $(0,1]$. Noting that $v(1) = u(1) = 0$, we see that v has at least n zeros in $(\tau, 1]$. Supposing that v does not have a zero in $(0,\tau)$ we can assume that $u, v > 0$ on $(0,\tau)$ with $u'(\tau) < 0$. Then

$$0 \geq A(\tau)u'(\tau)v(\tau) = \int_0^\tau \{Q(s) - q(s)\}u(s)v(s)ds > 0.$$

Hence we see that v has at least one zero in $(0,\tau)$ and so v has at least one more zero than u in $(0,1]$.

Corollary 4.2 *Let A be a profile with tapering of order $p \in [0,2)$ and let $u \not\equiv 0$ be a solution of Problem P with $\mu \leq \mu_i$. Then u has at most $i-1$ zeros in $(0,1]$ and $\lim_{s\to 0} u(s) = \eta$ exists. Furthermore η is finite and $\sin\eta \neq 0$. In particular, u is bounded and has a finite number of zeros for all $(\mu, u) \in E$.*

Remark From now on, when $p < 2$, the solutions of Problem P can be regarded as elements of $H_p \cap C([0,1])$.

Proof Setting

$$q(s) = \begin{cases} \mu\frac{\sin u(s)}{u(s)} & \text{if } u(s) \neq 0 \\ \mu & \text{if } u(s) = 0 \end{cases},$$

we see that u satisfies (4.3), and by Theorem 3.13, $u \in L^\infty(0,1)$. Clearly $q < \mu \leq \mu_i$ except at the zeros of u, and they form a set of measure zero in $(0,1]$. It follows from the Proposition 4.1 that the eigenfunction φ_i associated with the eigenvalue μ_i of the Problem PL has more zeros than u in $(0,1]$. Thus, by Theorem 2.5, u has at most $i-1$ zeros.

With Q defined by (3.22), we have that $Q \in L^1(0,1)$ and, integrating (1.13) twice, we find that, for $t \in (0,1]$,

$$u(t) = \mu\int_t^1 A(s)^{-1}\int_0^s \sin u(\sigma)d\sigma ds \tag{4.5}$$

where

$$\left|A(s)^{-1}\int_0^s \sin u(\sigma)d\sigma\right| \leq Q(s).$$

Hence, $\lim_{t\to 0} u(t)$ exists and

$$\lim_{t\to 0} u(t) = \eta \text{ where } \eta = \mu\int_0^1 A(s)^{-1}\int_0^s \sin u(\sigma)d\sigma ds.$$

Clearly,

$$|\eta| \leq \mu\int_0^1 Q(s)ds < \infty$$

and (4.5) yields

$$u(t) = \eta - \mu \int_0^t A(s)^{-1} \int_0^s \sin u(\sigma) d\sigma ds \tag{4.6}$$

for all $t > 0$. Suppose that $\sin \eta = 0$, and set $v(t) = \eta - u(t)$. Then (4.6) becomes

$$v(t) = -\mu \cos \eta \int_0^t A(s)^{-1} \int_0^s \sin v(\sigma) d\sigma ds$$

which implies that for all $s \in (0, 1]$,

$$|v(t)| \leq \mu \max_{0<s\leq t} |v(s)| \int_0^t Q(\sigma) d\sigma.$$

It follows that

$$\max_{0<t\leq T} |v(t)| \leq \mu \max_{0<t\leq T} |v(t)| \int_0^T Q(\sigma) d\sigma$$

for $0 < T \leq 1$. But $Q \in L^1(0,1)$ and so there must exist a $T > 0$ such that $\max_{0<t\leq T} |v(t)| = 0$. Then $u(T) = \eta$ and $u'(T) = 0$ and since u satisfies the differential equation (1.13) we conclude that $u \equiv \eta$ on $(0, 1]$. But $u(1) = 0$ so in fact $u \equiv 0$ on $(0, 1]$. Thus $\sin \eta \neq 0$ if $u \not\equiv 0$ on $(0, 1]$.

Theorem 4.3 *Let A be a profile with tapering of order $p \in [0, 2)$.*
(i) For every $i \in \mathbb{N}$, there are an open neighbourhood W of $(\mu_i, 0)$ in $\mathbb{R} \times H_A$ and two continuous functions

$$r : (-\delta, \delta) \to \mathbb{R} \text{ and } z : (-\delta, \delta) \to H_A$$

where $\delta > 0$ such that $r(0) = \mu_i, z(0) = 0$, $\langle \varphi_i, z(\xi) \rangle_A = 0$ for all ξ and

$$E \cap W = \{(r(\xi), \xi[\varphi_i + z(\xi)]) : 0 < |\xi| < \delta\}.$$

(ii) Furthermore, $z(\xi) \in L^\infty(0,1)$ and, for $\xi \neq 0$, $u_\xi \equiv \xi[\varphi_i + z(\xi)]$ has exactly i zeros in $[0, 1]$.
(iii) As $\xi \to 0$, $z'(\xi) \to 0$ uniformly on compact subsets of $(0, 1]$.
(iv) For all $\xi \neq 0, r(\xi) = r(-\xi) > \mu_i$ and $z(\xi) = z(-\xi)$.

Proof We apply the theorem on bifurcation from a simple eigenvalue, in the form due to Crandall and Rabinowitz (Theorem 1.7 of [10]) to the function F defined by (4.2). We already know from Lemma 2.12 that $F \in C^1(\mathbb{R} \times H_A, H_A)$ with $D_u F(\mu, u) = I - \mu L_A(u)$. Clearly $D_u F(\mu, u)$ is a smooth function of μ and $D_\mu D_u F = -G'_A = -L_A$ is continuous from $\mathbb{R} \times H_A$ into $B(H_A, H_A)$ by part (iii) of Lemma 2.12. In the notation of Section 3, $D_u F(\mu, 0) = I - \mu T$ so that, by Theorem 2.5, $\ker D_u F(\mu_i, 0) = \ker(I - \mu_i T) = span\,\{\varphi_i\}$ and $rge D_u F(\mu_i, 0) = \{\varphi_i\}^\perp$ since $T : H_A \to H_A$ is a compact self-adjoint operator. It follows that $D_\mu D_u F(\mu_i, 0)\varphi_i = -T\varphi_i = -\frac{1}{\mu_i}\varphi_i \notin rge D_u F(\mu_i, 0)$. This means that all of the hypotheses of Theorem 1.7 in [10] are satisfied. This proves part (i).

(ii) By Theorem 3.13, $u_\xi \in L^\infty(0,1)$ and by Theorem 2.5, $\varphi_i \in L^\infty(0,1)$. Consequently $z(\xi) \in L^\infty(0,1)$. By choosing δ small enough we may suppose that $r(\xi) < \mu_{i+1}$ for all $\xi \in (-\delta, \delta)$ and hence it follows from Corollary 4.2 that

u_ξ has at most i zeros in $(0,1]$ for $0<|\xi|<\delta$. On the other hand, by Theorem 3.13,

$$\|u_\xi\|_{L^\infty(0,1)} \le Dr(\xi)^\alpha \|u_\xi\|_A^\beta$$

and so $\|u_\xi\|_{L^\infty(0,1)} \to 0$ as $\xi \to 0$. Hence, by choosing δ small enough, we may assume that

$$r(\xi)\frac{\sin u_\xi(s)}{u_\xi(s)} > \mu_{i-1}$$

for all $s \in (0,1]$ such that $u_\xi(s) \neq 0$ whenever $0<|\xi|<\delta$. Setting

$$Q_\xi(s) = \begin{cases} r(\xi)\frac{\sin u_\xi(s)}{u_\xi(s)} & \text{if } u_\xi(s)\neq 0 \\ r(\xi) & \text{if } u_\xi(s) = 0 \end{cases},$$

we see that $v = u_\xi$ satisfies (4.4) where $Q_\xi(s) > \mu_{i-1}$ on $(0,1)$. It follows from Proposition 4.1 that u_ξ has more zeros than φ_{i-1} in $(0,1]$. Thus u_ξ has at least i zeros.

(iii) For $s \in (0,1]$, we have that

$$A(s)u_\xi'(s) = -r(\xi)\int_0^s \sin u_\xi(t)dt$$

and so, for $\xi \neq 0$,

$$\begin{aligned} A(s)[\varphi_i'(s) + z_\xi'(s)] = [\mu_i - r(\xi)] \int_0^s \frac{\sin u_\xi(t)}{u_\xi(t)}[\varphi_i(t) + z_\xi(t)]dt \\ - \mu_i \int_0^s \frac{\sin u_\xi(t)}{u_\xi(t)}[\varphi_i(t)+z_\xi(t)]dt. \end{aligned} \tag{4.7}$$

But

$$\left|\int_0^s \frac{\sin u_\xi(t)}{u_\xi(t)} z_\xi(t)dt\right| \le \int_0^s |z_\xi(t)|\,dt \le |z|_2 \le 2\,\|z_\xi\|_p$$

by Lemma 2.2, and $\|z_\xi\|_p \to 0$ as $\xi \to 0$ by the continuity of $z : (-\delta,\delta) \to H_A$ since $z_0 = 0$. Hence

$$\int_0^s \frac{\sin u_\xi(t)}{u_\xi(t)} z_\xi(t)dt \to 0 \text{ as } \xi \to 0,$$

uniformly with respect to s on $[0,1]$. Since $\frac{\sin u_\xi(t)}{u_\xi(t)} \to 1$ as $\xi \to 0$ uniformly on compact subsets of $(0,1]$ and $\int_0^s |\varphi_i(t)|\,dt \to 0$ as $s \to 0$, it follows from (4.7) that

$$A(s)[\varphi_i'(s) + z_\xi'(s)] \to -\mu_i \int_0^s \varphi_i(t)dt \text{ as } \xi \to 0$$

uniformly with respect to s on $[0,1]$. Noting that

$$-\mu_i\int_0^s \varphi_i(t)dt = A(s)\varphi_i'(s),$$

the proof of part (iii) is complete.

(iv) For ξ small enough, $(r(\xi), -u_\xi) \in E \cap W$ and so, by part (i) there exists $\eta \in (-\delta, \delta)$ such that $(r(\xi), -u_\xi) = (r(\eta), \eta[\varphi_i + z(\eta)])$. But this means that $-\xi[\varphi_i + z(\xi)] = \eta[\varphi_i + z(\eta)]$ and since $\langle \varphi_i, z(\xi)\rangle_A = \langle \varphi_i, z(\eta)\rangle_A = 0$, this implies that $\eta = -\xi$. Thus $r(\xi) = r(-\xi)$ and $z(\xi) = z(-\xi)$. Finally, using part (ii) and Corollary 4.2, we see that $r(\xi) > \mu_i$ for $\xi \neq 0$.

Lemma 4.4 *Let A be a profile with tapering of order $p \in [0, 2)$ and let (μ, u) be a solution of Problem P. Consider a sequence $\{(\mu_n, u_n)\}$ of solutions of Problem P such that $\mu_n \to \mu$ and $\|u - u_n\|_A \to 0$. Then*
(a) $\|u - u_n\|_{L^\infty(0,1)} \to 0$, and
(b) for any $\delta \in (0, 1), \|u - u_n\|_{C^1([\delta,1])} \to 0$.

Proof (a) If $u \equiv 0$, the result follows from Theorem 3.13. Suppose henceforth that $u \not\equiv 0$. For $s \in (0, 1]$, it follows from (4.5) that

$$\begin{aligned}
&|u_n(s) - u(s)| \\
&\leq |\mu - \mu_n| \int_s^1 A(\sigma)^{-1} \int_0^\sigma |\sin u_n(t)|\, dt d\sigma \\
&+ \mu \int_s^1 A(\sigma)^{-1} \int_0^\sigma |\sin u(t) - \sin u_n(t)|\, dt d\sigma \\
&\leq |\mu - \mu_n| \int_0^1 Q(\sigma) d\sigma + \mu \int_0^1 A(\sigma)^{-1} \int_0^\sigma |\sin u(t) - \sin u_n(t)|\, dt d\sigma
\end{aligned}$$

where Q is defined by (3.22). Since $Q \in L^1(0, 1)$ and $u_n \to u$ pointwise on $(0, 1]$, the Dominated Convergence Theorem implies that

$$\int_0^1 A(\sigma)^{-1} \int_0^\sigma |\sin u(t) - \sin u_n(t)|\, dt d\sigma \to 0$$

and the result follows immediately.

(b) But, if (λ, v) is a solution of Problem P,

$$\begin{aligned}
&A(s)\{u'(s) - v'(s)\} \\
&= -\mu \int_0^s \sin u(t) dt + \lambda \int_0^s \sin v(t) dt \\
&= (\lambda - \mu) \int_0^s \sin u(t) dt + \lambda \int_0^s \sin v(t) - \sin u(t) dt
\end{aligned}$$

and so, for $0 < \delta \leq s \leq 1$,

$$|u'(s) - v'(s)| \leq [K_2 \delta^p]^{-1} \{|(\lambda - \mu)| + |\lambda|\, \|u - v\|_{L^\infty(0,1)}\}.$$

Using part (a) it follows that $\|u - u_n\|_{C^1([\delta,1])} \to 0$.

Corollary 4.5 *Let A be a profile with tapering of order $p \in [0, 2)$ and let (μ, u) be a solution of Problem P such that u has exactly n zeros in $[0, 1]$. There is an open neighbourhood W of (μ, u) in $\mathbb{R} \times H_A$ such that v has also exactly n zeros in $[0, 1]$ for all solutions (λ, v) of Problem P in W.*

Proof By Corollary 4.2 we know that $\eta = \lim_{s \to 0} u(s) \neq 0$ and so by Lemma 4.4(a), there exist $\varepsilon, \delta > 0$ such that $v(s) \neq 0$ for all $s \in (0, \delta]$ and all solutions (λ, v) of Problem P such that $|\mu - \lambda| + \|u - v\|_A < \varepsilon$. Hence $u(\delta) \neq 0$ and u has exactly n zeros, all of which are simple, in $[\delta, 1]$. Consequently, there is an $\xi > 0$ such that v has exactly n zeros in $[\delta, 1]$ whenever $v \in C^1([\delta, 1])$ and $\|u - v\|_{C^1([\delta,1])} < \xi$. But by Lemma 4.4(b), there exists $\varepsilon_1 \in (0, \varepsilon)$ such that $\|u - v\|_{C^1([\delta,1])} < \xi$ for all solutions (λ, v) of Problem P with $|\mu - \lambda| + \|u - v\|_A < \varepsilon_1$ and consequently v has exactly n zeros in $[\delta, 1]$ for all these solutions. Since $\varepsilon_1 < \varepsilon$ these solutions have no zeros in $(0, \delta)$ and the proof is complete.

Fig. 9 Sotution of Problem P, $p = 1/2$ and $\mu = 2$

Fig. 10 Configuration of rod

We now come to the main results of this section. Given a profile A with tapering of order $p < 2$ and $i \in \mathbb{N}$, consider $E_i = E \cup \{(\mu_i, 0)\}$ with the metric inherited from $\mathbb{R} \times H_A$ and let C_i denote the maximal connected subset of this space which contains the point $(\mu_i, 0)$. Let

$$C_i^+ = \{(\mu, u) \in C_i : u'(1) < 0\} \text{ and let } C_i^- = \{(\mu, u) \in C_i : u'(1) > 0\}.$$

Clearly $C_i^+ \cap C_i^- = \emptyset$ and $C_i = C_i^+ \cup C_i^- \cup \{(\mu_i, 0)\}$.
We set $P(\mu, u) = \mu$.

Theorem 4.6 *Let A be a profile with tapering of order $p < 2$ and, for $i \in \mathbb{N}$, let C_i be the component of solutions of Problem P defined above. Then*
(a) C_i^+ is a connected subset of $\mathbb{R} \times H_A$ and of $\mathbb{R} \times C([0,1])$. Furthermore

$$C_i^- = \left\{(\mu, -u) : (\mu, u) \in C_i^+\right\} \text{ and } C_i^+ = \left\{(\mu, -u) : (\mu, u) \in C_i^-\right\}.$$

(b) $u(0) \neq 0$ *and* u *has exactly* i *zeros in* $[0,1]$ *for all* $(\mu, u) \in C_i^+$.
(c) $PC_i^+ = (\mu_i, \infty)$.
(d) $\|u\|_{L^\infty(0,1)} \in (0, \pi)$ *for all* $(\mu, u) \in C_i^+$.
(e) There is a function $U \in C^1((\mu_1, \infty), H_A)$ *such that*
$C_1^+ = \{(\mu, U(\mu)) : \mu_i < \mu < \infty\}$. *Furthermore* $U(\mu)$ *is the minimizer* u_μ *of the energy* J_μ *discussed in Theorem 3.3.*

Proof (a) We can suppose that the eigenfunction φ_i has $\varphi_i'(1) < 0$. By part (iii) of Theorem 4.3, we then have that $u_\xi'(1) < 0$ for all $\xi \in (0, \delta)$ and that $u_\xi'(1) > 0$ for all $\xi \in (-\delta, 0)$. Thus $u_\xi \in C_i^+$ if and only if $\xi \in (0, \delta)$.

To show that C_i^+ is a connected subset of $\mathbb{R} \times H_A$, we begin by recalling that a topological space X is disconnected if and only if there is a continuous function from X onto the two point subset $\{0, 1\}$ of $\mathbb{R}$. Let us suppose that C_i^+ is disconnected and that $f : C_i^+ \to \{0, 1\}$ is such a function. Since f is constant on the set $\{(r(\xi), u_\xi) : 0 < \xi < \delta\}$ by the continuity of $(r(\xi), u_\xi)$ as a function from $(0, \delta)$ into $\mathbb{R} \times H_A$, we can suppose that $f(r(\xi), u_\xi) = 0$ for all $\xi \in (0, \delta)$. Now extend f to all of C_i by setting $f(\mu, u) = 0$ for $(\mu, u) \in C_i \backslash C_i^+$. By Lemma 4.4(b), C_i^+ is an open subset of C_i and it follows easily that f is continuous on C_i, contradicting its connectedness. Hence C_i^+ is a connected subset of E_i for the $\mathbb{R} \times H_A$ topology.

It follows from this that $\left\{(\mu, -u) : (\mu, u) \in C_i^+\right\}$ is also a connected subset of E_i and

$$(r(-\xi), u_{-\xi}) = (r(\xi), -u_\xi) \in \left\{(\mu, -u) : (\mu, u) \in C_i^+\right\} \cap C_i$$

for $0 < \xi < \delta$. Therefore $\left\{(\mu, -u) : (\mu, u) \in C_i^+\right\} \cap C_i \neq \emptyset$ and so, by the maximality of C_i, $\left\{(\mu, -u) : (\mu, u) \in C_i^+\right\} \subset C_i$. Hence $\left\{(\mu, -u) : (\mu, u) \in C_i^+\right\} \subset C_i^-$. Interchanging the roles of C_i^+ and C_i^-, we see that $\left\{(\mu, -u) : (\mu, u) \in C_i^+\right\} = C_i^-$.

Now consider a non-empty subset V of C_i^+ which is both open and closed in C_i^+ for the metric of $\mathbb{R} \times C([0,1])$. If $(\lambda, v) \in V$, there exists $\delta > 0$ such that $(\mu, u) \in V$ for all $(\mu, u) \in C_i^+$ such that $|\mu - \lambda| + \|u - v\|_{L^\infty(0,1)} < \delta$. But, by Lemma 4.4(a), there exists $\delta_1 > 0$ such that $|\mu - \lambda| + \|u - v\|_{L^\infty(0,1)} < \delta$ whenever $(\mu, u) \in C_i^+$ and $|\mu - \lambda| + \|u - v\|_A < \delta_1$. Thus we see that V is also an open subset of C_i^+ for the metric of $\mathbb{R} \times H_A$. On the other hand, if $(\lambda, v) \in C_i^+$ and there exists a sequence $\{(\lambda_n, v_n\} \subset V$ such that $|\lambda - \lambda_n| + \|v - v_n\|_A \to 0$, then it follows from Lemma 4.4(a) that $|\lambda - \lambda_n| + \|v - v_n\|_{L^\infty(0,1)} \to 0$ and so $(\lambda, v) \in V$. This shows that V is a closed subset of C_i^+ for the metric of $\mathbb{R} \times H_A$. Since C_i^+ is connected for this metric we must have that $V = C_i^+$, showing that C_i^+ is also connected for the metric of $\mathbb{R} \times C([0,1])$.

(b) By Corollary 4.2, we can define an integer valued function N on E_i by setting $N((\mu_i, 0)) = i$ and $N((\mu, u)) = n(u)$ where $n(u)$ is the number of zeros of u in $(0, 1]$ if $(\mu, u) \in E$. Using Theorem 4.3 and Corollary 4.5, we see that N is locally constant on E_i and hence, from the connectedness of C_i, that $N((\mu, u)) = i$ for all $(\mu, u) \in C_i$. By Corollary 4.2, $u(0) \neq 0$ for all $(\mu, u) \in C_i^+$.

(c) We begin by applying the well-known global bifurcation theorem of Rabinowitz, [22], to the equation $F(\mu, u) = 0$ where F is defined by (4.2). We know from Lemma 2.12 that $F \in C^1(\mathbb{R} \times H_A, H_A)$ with $D_u F(\mu, 0) = I - \mu T$. The compactness of $G_A : H_A \to H_A$ and $T = L_A(0) : H_A \to H_A$ is established

in Lemma 2.11 and Lemma 2.12(ii), respectively. Furthermore, by Theorem 2.5, $\ker D_u F(\mu_i, 0) = \ker(I - \mu_i T) = span\,\{\varphi_i\}$ and $rge D_u F(\mu_i, 0) = \{\varphi_i\}^{\perp}$ since $T : H_A \to H_A$ is a compact self-adjoint operator. It follows that C_i has at least one of the following properties.
(1) It is an unbounded subset of $\mathbb{R} \times H_A$.
(2) Its closure in $\mathbb{R} \times H_A$ contains a point $(\mu_j, 0)$ with $j \neq i$.

Using Theorem 4.3 and part (b) it follows that C_i cannot have the property (2). Then property (1) and (3.24) imply that PC_i is unbounded. But, part (b) and Corollary 4.2 show that $P[C_i \backslash \{(\mu_i, 0)\}] \subset (\mu_i, \infty)$, so in fact $PC_i^+ = P[C_i \backslash \{(\mu_i, 0)\}] = (\mu_i, \infty)$ by part (a).

(d) Let $A = \left\{(\mu, u) \in C_i : \|u\|_{L^\infty(0,1)} < \pi\right\}$. Clearly $(\mu_i, 0) \in A$ and by Lemma 4.4(a), A is an open subset of C_i. On the other hand, again by Lemma 4.4(a), $\|u\|_{L^\infty(0,1)} \leq \pi$ if (μ, u) belongs to the closure of A in C_i. But then $|u(s)| < \pi$ for all $s \in (0, 1]$ by Theorem 3.5(i). Hence, either $\|u\|_{L^\infty(0,1)} < \pi$ or $\lim_{s\to 0} u(s) = \pm\pi = \pm \|u\|_{L^\infty(0,1)}$. But Corollary 4.2 shows that the latter case cannot occur. Hence $\|u\|_{L^\infty(0,1)} < \pi$, proving that A is a closed subset of C_i. From the connectedness of C_i, it now follows that $A = C_i$.

(e) For $\mu > \Lambda(A) = \mu_1$, consider the energy minimizer u_μ. By Theorem 3.3 we know that $u_\mu(s) > 0$ for $s < 1$. By parts (b),(c) and (d), there is a element $(\mu, w) \in C_1^+$ and $0 < w(s) < \pi$ on $(0, 1)$. Corollary 3.4 shows that $w = u_\mu$ and hence, in the notation of Theorem 3.3(vi), $C_1^+ = \{(\mu, U(\mu)) : \mu_i < \mu < \infty\}$. To establish the additional regularity of this parametrization we shall apply the Implicit Function Theorem to $F \in C^1(\mathbb{R} \times H_A, H_A)$ at (μ, u_μ). We have that $F(\mu, u_\mu) = 0$. Also $D_u F(\mu, u_\mu)v = 0$ if and only if $v \in H_A$ and $v = \mu L_A(u_\mu)v$ where L_A is defined by (2.18). As in the proof of Theorem 3.1, this implies that $v \in C^1((0, 1]), Av' \in C^1((0, 1])$ and

$$\{A(s)v'(s)\}' + \mu \cos u_\mu(s)v(s) = 0 \text{ for } 0 < s < 1$$

with $0 = v(1) = \lim_{s\to 0} A(s)v'(s)$. Since $0 < u_\mu(s) < \pi$ for $s \in (0, 1)$, it follows that $\cos u_\mu(s) < \frac{\sin u_\mu(s)}{u_\mu(s)}$ on $(0, 1)$ and so by Proposition 4.1, either $v \equiv 0$ or u_μ has more zeros than v in $(0, 1]$. Since u_μ has exactly one zero and $v(1) = 0$, the second alternative is excluded and we have shown that $v \equiv 0$. From the compactness of $L_A(u_\mu) : H_A \to H_A$ that was established in Lemma 2.12, this implies that $D_u F(\mu, u_\mu) = I - \mu L_A(u_\mu) : H_A \to H_A$ is an isomorphism. By the Implicit Function Theorem there are a number $\delta > 0$, a C^1-function $\Psi : (\mu - \delta, \mu + \delta) \to H_A$ and an open neighbourhood W of (μ, u_μ) in $\mathbb{R} \times H_A$ such that $W \cap E = \{(\lambda, \Psi(\lambda)) : \lambda \in (\mu - \delta, \mu + \delta)\}$, $\Psi(\mu) = u_\mu$ and $0 \notin W$. Since (μ, u_μ) belongs to the connected set $C_1^+ = \{(\lambda, U(\lambda)) : \lambda \in (\mu_1, \infty)\}$, it follows that $U(\lambda) = \Psi(\lambda)$ for all $\lambda \in (\mu - \delta, \mu + \delta)$ and so $U \in C^1((\mu_1, \infty), H_A)$.

This completes the proof.

5 Bifurcation for critical tapering $(p = 2)$

In this section we give a more detailed discussion of the bifurcation diagram for Problem P in the critical case of tapering of order 2. We begin by recalling from Theorem 2.7 that, in this case, Problem PL has a non-empty essential spectrum.

Indeed,

$$0 < \Lambda(A) \leq \Lambda_e(A) = \frac{L}{4}$$

for a profile with tapering of order 2 with $L = \lim_{s\to 0} A(s)/s^2$. Thus two very different situations can occur, namely,
(1) $\Lambda(A) < \Lambda_e(A)$ and (2) $\Lambda(A) = \Lambda_e(A)$,

For profiles with tapering of order $p = 2$ the relationship between Problem P and its linearization Problem PL is weaker than when $p < 2$ since the function $F : \mathbb{R} \times H_A \to H_A$ defined by (4.2) is not Fréchet differentiable at $(\mu, 0)$ for any $\mu > 0$, by the remark following Lemma 2.12. As a first example of this discrepancy, let us discuss the nodal structure of solutions of Problem P.

In Corollary 4.2 it is shown that for profiles with tapering of order $p < 2$, non-trivial solutions of Problem P have only a finite number of zeros in $[0, 1]$ and this was obtained by comparison with the linearized Problem PL which has a similar property. Now for $p = 2$, we know from Lemma 2.6 that all solutions of Problem PL for $\mu > \frac{L}{4}$ have an infinite number of zeros. Nonetheless, non-trivial solutions of Problem P still have only a finite number of zeros in $[0, 1]$ as we now show. Clearly we cannot resort to comparison with the linearized equation and, so far, we have only been able to prove the result for profiles which satisfy an additional, but natural, regularity condition as $s \to 0$.

Theorem 5.1 *Let A be a profile with tapering of order 2 which is differentiable near 0 and such that $\lim_{s\to 0} A'(s)/s$ exists. If u is a non-trivial solution of Problem P, u has only a finite number of zeros in* $[0, 1]$.
If, in addition, A has the property (3.15) then $\|u\|_{L^\infty(0,1)} = \pi$ and $\lim_{s\to 0} u(s) = \pm\pi$.

Remark By L'Hospital's rule we see that

$$\lim_{s\to 0} A'(s)/s = 2L \text{ where } L = \lim_{s\to 0} \frac{A(s)}{s^2} \tag{5.1}$$

and hence

$$\lim_{s\to 0} \frac{A'(s)}{\sqrt{A(s)}} = 2\sqrt{L}. \tag{5.2}$$

For the expression of the above hypotheses on A in terms of the physical variables (1.10) and (1.11) for the buckling of a rod under its own weight, see the remark following Theorem 3.5.

Proof Let $\delta > 0$ be such that A is differentiable on $[0, \delta]$ and consider the functions V and W defined by (3.16) and (3.17). By reducing δ if necessary, we may suppose that $A'(s) \geq Ls > 0$ and that

$$A'(s) \geq \frac{1}{2}\sqrt{LA(s)} \text{ on } (0, \delta]. \tag{5.3}$$

Consider a non-trivial solution u of Problem P for some $\mu > 0$. Clearly u has only a finite number of zeros in any compact subset of $(0, 1]$. Let us suppose that u has an infinite number of zeros $\{z_n : n \in \mathbb{N}\}$ in $(0, \delta)$ where $z_{n+1} < z_n$. Then $\lim_{n\to\infty} z_n = 0$ and $u'(z_n) \neq 0$.

Using the functions V as in the proof of Theorem 3.5(ii) (but only on the interval $(0,\delta]$), we see that $|u(s)| < \pi$ for all $s \in (0,\delta]$ and, since $\{A(s)u'(s)\}' = -\mu \sin u(s)$, this implies that $A(s)u'(s)$ is strictly monotone on (z_{n+1}, z_n). Hence u' has exactly one zero, which we denote by t_n, in the interval (z_{n+1}, z_n). Now using the function W as in the proof of Theorem 3.5(ii), we see that $W' < 0$ on $(0,\delta]\backslash \{t_n : n \in \mathbb{N}\}$ and $|u(t_n)| < |u(t_{n+1})| < \pi$ for all $n \in \mathbb{N}$. Let $U = \lim_{n\to\infty} |u(t_n)|$. Furthermore,

$$\lim_{s\to 0} W(s) = \lim_{t_n\to 0} W(t_n) = \lim_{t_n\to 0} \mu\{1-\cos u_n\} = \mu\{1-\cos U\} \leq 2\mu$$

where $u_n = u(t_n)$. Thus

$$\frac{1}{2}A(s)u'(s)^2 = \frac{1}{2A(s)}[A(s)u'(s)]^2 \leq W(s) \leq 2\mu \text{ for all } s \in (0,\delta]. \tag{5.4}$$

We shall now show that there is a number $\Delta > 0$ such that $W(t_{n+1}) - W(t_n) \geq \Delta$ for all $n \in \mathbb{N}$. This is incompatible with the boundedness of W as $s \to 0$ and so the existence of infinitely many zeros of u will be excluded. Replacing u by $-u$ if necessary, we can suppose that $u_{n+1} < 0 < u_n$ and that $u' > 0$ on (t_{n+1}, t_n).

Furthermore,

$$\begin{aligned}\int_{t_{n+1}}^{t_n} A(s)u'(s)u(s)\left(\frac{1}{\sqrt{A(s)}}\right)' ds &= -\frac{1}{2}\int_{u_{n+1}}^{u_n} \frac{A'(s)}{\sqrt{A(s)}} u\,du \\ &= -\frac{1}{2}\int_{u_{n+1}}^{u_n} \left\{\frac{A'(s)}{\sqrt{A(s)}} - 2\sqrt{L}\right\} u\,du - \sqrt{L}\int_{u_{n+1}}^{u_n} u\,du\end{aligned}$$

and so, by (5.2),

$$\left|\int_{t_{n+1}}^{t_n} A(s)u'(s)u(s)\left(\frac{1}{\sqrt{A(s)}}\right)' ds\right| \leq \pi^2 \sup_{0<s\leq t_n} \left|\frac{A'(s)}{\sqrt{A(s)}} - 2\sqrt{L}\right| + \frac{\sqrt{L}}{2}\left\{u_{n+1}^2 - u_n^2\right\}$$

since $|u(s)| < \pi$ for all $s \in (0,\delta]$. Hence, by (5.2) and the fact that $|u_n| \to U$, we have that

$$\int_{t_{n+1}}^{t_n} A(s)u'(s)u(s)\left(\frac{1}{\sqrt{A(s)}}\right)' ds \to 0 \text{ as } n \to \infty. \tag{5.5}$$

Now, using (3.18) and (5.3),

$$\begin{aligned}W(t_{n+1}) - W(t_n) &= -\int_{t_{n+1}}^{t_n} W'(s)ds = \frac{1}{2}\int_{t_{n+1}}^{t_n} A'(s)u'(s)^2 ds \\ &\geq \frac{\sqrt{L}}{4}\int_{t_{n+1}}^{t_n} \sqrt{A(s)}u'(s)^2 ds.\end{aligned}$$

But,

$$\int_{t_{n+1}}^{t_n} \sqrt{A(s)}u'(s)^2 ds = \int_{t_{n+1}}^{t_n} \frac{A(s)}{\sqrt{A(s)}}u'(s)^2 ds$$
$$= -\int_{t_{n+1}}^{t_n} \frac{\{Au'\}' u}{\sqrt{A}} + \{Au'\} u \left(\frac{1}{\sqrt{A}}\right)' ds$$
$$= \mu \int_{t_{n+1}}^{t_n} \frac{1}{\sqrt{A}} u \sin u ds - \int_{t_{n+1}}^{t_n} \{Au'\} u \left(\frac{1}{\sqrt{A}}\right)' ds$$

and, by (5.4),

$$\int_{t_{n+1}}^{t_n} \frac{1}{\sqrt{A}} u \sin u ds \geq \frac{1}{2\sqrt{\mu}} \int_{t_{n+1}}^{t_n} u' u \sin u ds$$
$$= \frac{1}{2\sqrt{\mu}} \int_{u_{n+1}}^{u_n} u \sin u du \geq \frac{1}{\sqrt{\mu}} \int_0^{u_n} u \sin u du$$
$$\geq 2D$$

where $D = \frac{1}{2\sqrt{\mu}} \int_0^{|u_1|} u \sin u du$ since $u_{n+1} \leq -u_n < 0 < u_n$ and $u_n \geq |u_1|$. Recalling (5.5), we see that there exists N such that

$$W(t_{n+1}) - W(t_n) \geq \frac{\sqrt{L}}{4} \int_{t_{n+1}}^{t_n} \sqrt{A}[u']^2 ds \geq \frac{\sqrt{L}\mu D}{4} = \Delta \text{ for all } n \geq N.$$

Hence $W(t_{N+k}) \geq W(t_N) + k\Delta$ for all $k \in \mathbb{N}$, contradicting (5.4). Thus u has only a finite number of zeros in $[0,1]$.

Suppose now that A has also the property (3.15) and let z be the smallest zero of u in $(0,1]$. By Theorem 3.5 we have that $|u(s)| < \pi$ for all $s \in (0,1]$. Replacing u by $-u$ if necessary we can assume that $0 < u(s) < \pi$ for all $s \in (0,z)$. The proof of Theorem 3.10(i) now shows that $\lim_{s\to 0} u(s) = \pi$.

This completes the proof of the theorem.

Fig. 11 Solution of Problem P, $p = 2$ and $\mu = 15$

Fig. 12 Configuration of the rod

Fig. 13 Solution of Problem P, $p = 2$ and $\mu = 15$

Fig. 14 Configuration of the rod

5.1 The case $\Lambda(A) < \Lambda_e(A)$

In this case, $\Lambda(A)$ is an isolated eigenvalue of multiplicity one and it might seem that, at least locally, the situation concerning bifurcation at $\Lambda(A)$ is exactly the same as in Theorem 4.3 for profiles with tapering of order $p < 2$. However, since $p = 2$, we know from the remark following Lemma 2.12 that the function $F : \mathbb{R} \times H_A \to H_A$ defined by (4.2) is not Fréchet differentiable at $(\mu, 0)$ for any $\mu > 0$ and so the Crandall-Rabinowitz bifurcation theorem cannot be used. Nonetheless we already know from Theorem 3.3 that a branch of solutions $\{(\mu, u_\mu) : \mu > \Lambda(A)\}$ bifurcates at $\Lambda(A)$. These solutions are positive on $[0, 1)$ and, by Corollary 3.4(i), any non-trivial solution (μ, u) of Problem P with $u \neq \pm u_\mu$ and $|u| \leq \pi$ on $(0, 1]$ has at least one zero in $(0, 1)$. Let

$$\lambda^* = \max \sigma(T) \backslash \lambda_1 \text{ and set } \Lambda_2(A) = \frac{1}{\lambda^*} \tag{5.6}$$

where $T : H_A \to H_A$ is the bounded self-adjoint operator defined by (2.7) and $\sigma(T)$ is its spectrum with $\lambda_1 = \frac{1}{\Lambda(A)} = \max \sigma(T)$. Then λ_1 is a simple eigenvalue of T by Lemma 2.4 and $\lambda_1 > \lambda^* \geq \max \sigma_e(T) = \frac{4}{L}$. Thus

$$\Lambda(A) < \Lambda_2(A) = \inf \sigma(A) \backslash \Lambda(A) \leq \Lambda_e(A). \tag{5.7}$$

We shall now show $\pm u_\mu$ are the only non-trivial solutions of Problem P with for $|u| \leq \pi$ on $(0, 1]$ for $\mu \in (\Lambda(A), \Lambda_2(A))$. For this we shall use the minimax principle for self-adjoint operators, (see Chapter XI.1 of [11]) which we now recall in a suitable notation. For a bounded self-adjoint operator S acting on a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$, let

$$\Gamma(S) = \sup\{\langle Su, u\rangle : u \in H \text{ with } \|u\| = 1\} \tag{5.8}$$

and

$$\gamma(S) = \inf_{v\in H\setminus\{0\}} \sup\{\langle Su, u\rangle : u \in H \text{ with } \|u\| = 1 \text{ and } \langle u, v\rangle = 0\}. \tag{5.9}$$

Then $\Gamma(S) = \max\sigma(S)$ and, if $\Gamma(S) > \max\sigma_e(S)$, then $\Gamma(S) = \max\sigma(S)$ is an eigenvalue of S. If, in addition, $\Gamma(S)$ is a simple eigenvalue of S then $\gamma(S) = \max\sigma(S)\backslash\Gamma(S) < \Gamma(S)$. Clearly if $S_1 \le S_2$, then $\Gamma(S_1) \le \Gamma(S_2)$ and $\gamma(S_1) \le \gamma(S_2)$.

For $u, v, w \in H_A = H_2$, let

$$B(u)(v,w) = \int_0^1 \frac{\sin u(s)}{u(s)} v(s)w(s)ds$$

where $\frac{\sin u(s)}{u(s)}$ is interpreted as 1 when $u(s) = 0$. As in Lemma 2.12(i), it follows from Lemma 2.2 that $B(u) : H_A \times H_A \to \mathbb{R}$ is a bounded symmetric bilinear form and so there is a unique bounded self-adjoint linear operator $T(u) : H_A \to H_A$ such that

$$\langle T(u)v, w\rangle_A = \int_0^1 \frac{\sin u(s)}{u(s)} v(s)w(s)ds$$

for all $u, v, w \in H_A$. Furthermore $T(0)$ is equal to the operator T discussed above and

$$\langle T(u)v, v\rangle_A \le \langle T(0)v, v\rangle_A$$

for all $u, v \in H_A$. Hence $T(u) \le T$ for all $u \in H_A$. This implies that

$$\max\sigma(T(u)) \le \max\sigma(T) = \Gamma(T) = \frac{1}{\Lambda(A)}$$
$$\text{and } \max\sigma_e(T(u)) \le \max\sigma_e(T) \le \lambda^* = \gamma(T) = \frac{1}{\Lambda_2(A)}$$

for all $u \in H_A$ since we know from Lemma 2.4 that $\frac{1}{\Lambda(A)}$ is a simple eigenvalue of T.

Theorem 5.2 *Let A be a profile with tapering of order 2 such that $\Lambda(A) < \Lambda_e(A)$. Suppose that (μ, u) is a non-trivial solution of Problem P with $|u| \le \pi$ on $(0,1]$ and $\Lambda(A) < \mu < \Lambda_2(A)$. Then $u = \pm u_\mu$.*

Remark Recall from Theorem 3.5(ii) that all solutions of Problem P satisfy $|u| \le \pi$ on $(0,1]$ under a weak monotonicity assumption (3.15) on the profile A.

Proof As in the proof of Lemma 2.4, it follows that u is an eigenfunction of $T(u)$ with eigenvalue $\lambda = \frac{1}{\mu} > \gamma(T) \ge \gamma(T(u))$ and that λ is a simple eigenvalue of $T(u)$. Thus $\lambda = \Gamma(T(u)) > \gamma(T(u))$ and

$$\frac{\langle T(u)u, u\rangle_A}{\langle u, u\rangle_A} = \lambda = \sup\{\langle T(u)v, v\rangle_A : v \in H \text{ with } \|v\|_A = 1\}.$$

But $|u| \in H_A$ with $\langle |u|, |u| \rangle_A = \langle u, u \rangle_A$ and $\langle T(u)|u|, |u| \rangle_A = \langle T(u)u, u \rangle_A$ so

$$\frac{\langle T(u)\,|u|\,, |u| \rangle_A}{\langle |u|\,, |u| \rangle_A} = \sup \left\{ \langle T(u)v, v \rangle_A : v \in H \text{ with } \|v\|_A = 1 \right\}.$$

Hence $|u|$ must be an eigenfunction of $T(u)$ with eigenvalue $\Gamma(T(u))$. Since we know that $\Gamma(T(u))$ is a simple eigenvalue there is a constant α such that $|u| = \alpha u$ showing that u cannot change sign on $(0,1]$. It now follows from Corollary 3.4 that $u = \pm u_\mu$.

Remark The above result shows that the stable equilibria $\pm u_\mu$ are the only non-trivial solutions of Problem P for values of $\mu < \Lambda_2(A)$. If $\Lambda_2(A) < \Lambda_e(A)$, then $1/\Lambda_2(A)$ is a simple eigenvalue of T and bifurcation from $u = 0$ should occur. In fact, there should be bifurcation at $\mu_i = 1/\lambda_i$ for every $\lambda_i \in \sigma(T) \cap (\sigma_e(T), \|T\|]$. Since the function F defined by (4.2) is not Fréchet differentiable at $u = 0$, this does not follow from standard results such as [10], but an obvious variant of the approach used below to deal with bifurcation at every $\mu \geq \Lambda_e(A)$ should work. So far we have not explored this possibility.

5.2 Bifurcation at every $\mu \geq \Lambda_e(A)$

We now show that in both cases (1) and (2) there are infinitely many distinct solutions of Problem P for every $\mu > \Lambda_e(A)$ and that these solutions converge strongly to 0 in H_A. Thus every $\mu \geq \Lambda_e(A)$ is a bifurcation point for buckled equilibrium configurations. Note that in case (2), the linearized Problem PL may have no non-trivial solutions.

To deal with this situation we use a well-known result due to Clark, [7], based on the notion of the genus of a set, concerning the existence of an infinite number of critical points of a C^1−functional on a real Hilbert space $(H, \langle \cdot, \cdot \rangle)$. (See also [16] and [27].) Let

$$\Sigma = \{\Omega \subset H : \Omega \text{ is closed and } \Omega = -\Omega\}$$

and define the genus $g : \Sigma \to \mathbb{N} \cup \{0, \infty\}$ as follows :

$g(\emptyset) = 0$,

$g(\Omega) = k$ if there is an odd mapping $h \in C(\Omega, \mathbb{R}^k \backslash \{0\})$ and k is the smallest integer with this property, and

$g(\Omega) = \infty$ if there is no integer k with the above property.

Set

$$G_k = \{\Omega \in \Sigma : g(\Omega) \geq k\}.$$

Recalling that $g(\Omega) = k$ provided that there is an odd homeomorphism from Ω onto the unit sphere on $\mathbb{R}^k$, we have that $G_k \neq \emptyset$ for all $k \in \mathbb{N}$ when $\dim H = \infty$.

Following Palais and Smale, a functional $f \in C^1(H, \mathbb{R})$ is said to satisfy the condition (PS) on H provided that every sequence $\{w_n\} \subset H$ which has the properties :

(i) $\{f(w_n)\}$ is a bounded sequence and (ii) $\|\nabla f(w_n)\| \to 0$,

has a subsequence converging in H.

Theorem 5.3 *Let $f \in C^1(H, \mathbb{R})$ be an even functional with $f(0) = 0$ which is bounded below and satisfies the condition (PS). Suppose that $\dim H = \infty$ and* *that* $-\infty < b_k < 0$ *for all* $k \in \mathbb{N}$ *where*

$$b_k = \inf_{\Omega \in G_k} \sup_{w \in \Omega} f(w).$$

Setting $K_b = \{w \in H : f(w) = b$ *and* $\nabla f(w) = 0\}$, *we have that* $K_{b_k} \neq \emptyset$ *for all* $k \in \mathbb{N}$ *and that* $g(K_{b_k}) \geq j$ *if* $b_k = b_{k+1} = \ldots = b_{k+j-1}$. *In particular,* f *has an infinite number of critical points. Furthermore,* $\lim_{k\to\infty} b_k = 0$.

This result is due to Clark, [7], except for the conclusion that $\lim_{k\to\infty} b_k = 0$ which was established by Heinz, [16].

We begin by showing that the hypotheses of this result are satisfied by a modified version of the functional $J_\mu : H_A \to \mathbb{R}$ for any $\mu > \Lambda_e(A)$. The modification is made so that the solutions to Problem P which we obtain via Theorem 5.3 satisfy the additional condition that $|u(s)| < \pi$ for all $s \in (0, 1]$. If the profile A has the weak monotonicity property (3.15), we know by Theorem 3.5 that all solutions of Problem P are bounded by π and so the modification of J_μ is unnecessary.

Set

$$h(\theta) = \begin{cases} \sin\theta & \text{for } \theta \in [-\pi, \pi] \\ 0 & \text{for } \theta \notin [-\pi, \pi] \end{cases}$$

and let

$$H(\theta) = \int_0^\theta h(\sigma)d\sigma \text{ for all } \theta \in \mathbb{R}.$$

Clearly h is Lipschitz continuous on $\mathbb{R}$ with Lipschitz constant 1 and $H \in C^1(\mathbb{R})$ is even. In fact, $H(\theta) = 1 - \cos\theta$ for $\theta \in [-\pi, \pi]$ and $H(\theta) = 2$ for $\theta \notin [-\pi, \pi]$. For a profile A with tapering of order 2, we define new functionals φ and $j_\mu(u) : H_A \to \mathbb{R}$ by

$$\varphi(u) = \int_0^1 H(u(s))ds \text{ and } j_\mu(u) = \frac{1}{2}\|u\|_A^2 - \mu\varphi(u).$$

For $u, v \in H_A$,

$$\left|\int_0^1 v(s)h(u(s))ds\right| \leq \int_0^1 |v(s)|\,|u(s)|\,ds \leq 4\|u\|_2\,\|v\|_2$$

by Lemma 2.2, and so there is a unique element $D_A(u) \in H_A = H_2$ such that

$$\langle D_A(u), v\rangle_A = \int_0^1 v(s)h(u(s))ds$$

for all $v \in H_A$.

Lemma 5.4 *Let* A *be a profile with tapering of order* 2. *The functional* $\varphi : H_A \to \mathbb{R}$ *has the following properties.*
(i) $0 \leq \varphi(u) = \varphi(-u) \leq 2$ *for all* $u \in H_A$.
(ii) $\varphi \in C^1(H_A)$ *and* $\nabla\varphi = D_A$.
(iii) $\varphi : H_A \to \mathbb{R}$ *is weakly sequentially continuous and* $D_A : H_A \to H_A$ *is completely continuous.*

Proof The proofs of these properties are very similar to the analogous results for ψ and G_A in Section 2, so we only give some brief indication of the changes required for (ii). Note that

$$\begin{aligned} H(\theta+\eta) - H(\theta) - h(\theta)\eta &= \int_0^1 \frac{d}{dt} H(\theta + t\eta)dt - h(\theta)\eta \\ &= \int_0^1 \{h(\theta+t\eta) - h(\theta)\}\, \eta dt \end{aligned}$$

so that

$$|H(\theta+\eta) - H(\theta) - h(\theta)\eta| \leq \frac{\eta^2}{2} \text{ for all } \theta, \eta \in \mathbb{R},$$

by the Lipschitz continuity of h. Hence, for all $u, v \in H_A$,

$$|\varphi(u+v) - \varphi(u) - \langle D_A(u), v\rangle_A| \leq \frac{1}{2}\int_0^1 v(s)^2 ds \leq 2\, \|v\|_2^2\,,$$

showing that $\varphi'(u)v = \langle D_A(u), v\rangle_A$. Then, for all $u, v, w \in H_A$,

$$\begin{aligned} |\varphi'(u)v - \varphi'(w)v| &\leq \int_0^1 |v(s)|\, |h(u(s)) - h(w(s))|\, ds \\ &\leq \int_0^1 |v(s)|\, |u(s) - w(s)|\, ds \\ &\leq 4\, \|v\|_2\, \|u-w\|_2 \end{aligned}$$

and so $\varphi \in C^1(H_A)$.

Corollary 5.5 *Let A be a profile with tapering of order 2. For all $\mu > 0$, the functional $j_\mu : H_A \to \mathbb{R}$ has the following properties.*
(i) $j_\mu \in C^1(H_A)$ and $\nabla j_\mu = I - \mu D_A$.
(ii) j_μ is bounded below and satisfies the condition (PS).

Proof By Lemma 5.4, $j_\mu \in C^1(H_A)$ and $\nabla j_\mu(u) = I - \mu D_A$. Consider a sequence $\{w_n\} \subset H_A$ such that (i) $\{j_\mu(w_n)\}$ is bounded and (ii) $\|\nabla j_\mu(w_n)\|_A \to 0$. Since $j_\mu(u) = \frac{1}{2}\|u\|_A^2 - \mu\varphi(u)$ and $0 \leq \varphi(u) \leq 2$ for all $u \in H_A$, it follows immediately from (i) that $\{w_n\}$ is a bounded sequence in H_A. Passing to a subsequence we can suppose that $w_n \rightharpoonup w$ weakly in H_A and hence that $\|D_A(w_n) - D_A(w)\|_A \to 0$ by Lemma 5.4(iii). But then,

$$w_n = \nabla j_\mu(w_n) + \mu D_A(w_n) \to \mu D_A(w),$$

proving that the condition (PS) is satisfied. Clearly $j_\mu(u) \geq -2\mu$ for all $u \in H_A$.

We have introduced this modified energy functional because its stationary points have the following property.

Lemma 5.6 *Let A be a profile with tapering of order 2 and suppose that $\nabla j_\mu(u) = 0$ for some $\mu > 0$ and $u \in H_A$. Then u is a solution of Problem P and $|u(s)| < \pi$ for all $s \in (0,1]$.*

Proof Suppose that $u \in H_A$ and $\nabla j_\mu(u) = 0$. As in the proof of Theorem 3.1, we find that $u \in C^1((0,1]), Au' \in C^1((0,1])$ and

$$\{A(s)u'(s)\}' + \mu h(u(s)) = 0 \text{ for all } s \in (0,1] \tag{5.10}$$

with $\lim_{s\to 0} A(s)u'(s) = 0$ and $u(1) = 0$.

Suppose that there is a point $s_0 \in (0,1)$ such that $u(s_0) > \pi$ and let (a,b) be a maximal interval on which $u > \pi$. Then $h(u(s)) = 0$ on (a,b) and so there is a constant c such that $A(s)u'(s) = c$ on (a,b). Since $u(1) = 0$ we must have $c < 0$ and this implies that $a = 0$. But then $\lim_{s\to 0} A(s)u'(s) = c \neq 0$ which contradicts an earlier assertion. Hence $u \leq \pi$ on $(0,1]$. Now if there is a point $s \in (0,1)$ such that $u(s) = \pi$, $u'(s) = 0$ and consequently, $u \equiv \pi$ on $(0,1]$ by the uniqueness of the Cauchy problem for the equation (5.10). This is again a contradiction so we can conclude that $u(s) < \pi$ on $(0,1]$. Replacing u by $-u$ we see that $|u(s)| < \pi$ for all $s \in (0,1]$, and (5.10) becomes (1.13), as required.

In order to apply Theorem 5.3 to our problem we still need to estimate quantities b_k for the functional j_μ and for this the following result is crucial.

Lemma 5.7 *Let A be a profile with tapering of order 2. Given any $k \in \mathbb{N}$ and any $\varepsilon > 0$, there is a subspace E of $H_A \cap L^\infty(0,1)$ such that* $\dim E = k$ *and*

$$\int_0^1 A(s)u'(s)^2 ds \leq \{\Lambda_e(A) + \varepsilon\} \int_0^1 u(s)^2 ds$$

for all $u \in E$.

Proof Set $\alpha = 2\Lambda_e(A)^2$ and then fix $k \in \mathbb{N}$ and $\varepsilon \in (0, \Lambda_e(A))$. By Theorem 1.2(ii) of Chapter XI in [11], there is a subspace F of H_A such that $\dim F = k$ and

$$\langle Tu, u\rangle_A \geq \{\max \sigma_e(T) - \frac{\varepsilon}{4\alpha}\} \langle u, u\rangle_A$$

for all $u \in F$. Let $\{w_i : i = 1, ..., k\}$ be an orthonormal basis for F. But we observed in Proposition 2.1(iii) that $H_A \cap L^\infty(0,1)$ is dense in H_A. The continuity of $T : H_A \to H_A$ implies that there exist $\{v_i : i = 1, ..., k\} \subset H_A \cap L^\infty(0,1)$ such that $\dim E = k$ and

$$\langle Tu, u\rangle_A \geq \{\max \sigma_e(T) - \frac{\varepsilon}{2\alpha}\} \langle u, u\rangle_A$$

for all $u \in E$ where $E = span\,\{v_i : i = 1, ..., k\}$. The result follows since $\{\max \sigma_e(T) - \frac{\varepsilon}{2\alpha}\}^{-1} \leq \Lambda_e(A) + \varepsilon$.

Theorem 5.8 *Let A be a profile with tapering of order 2 and consider $\mu > \Lambda_e(A)$. For this value of μ, there are infinitely many solutions $\{u_k\}$of Problem P with the property that $|u_k(s)| < \pi$ for all $s \in (0,1]$. Furthermore, $\|u_k\|_A \to 0$ as $k \to \infty$ and the number of zeros of u_k tends to infinity as $k \to \infty$.*

Remark 4 If the profile A is differentiable near 0 and $\lim_{s\to 0} A'(s)/s$ exists we know from Theorem 5.1 that all non-trivial solutions of Problem P have only a finite number of zeros. If, in addition, A has the property (3.15) then Theorem 5.1 also shows that $\|u_k\|_{L^\infty(0,1)} = \pi$ for all k.

Proof To establish the existence of a sequence of solutions we apply Theorem 5.3 to the functional $j_\mu : H_A \to \mathbb{R}$. In view of Corollary 5.5 and Lemma 5.6, we only need to show that $b_k < 0$ for all $k \in \mathbb{N}$. To this end we choose $k \in \mathbb{N}$ and $\varepsilon > 0$ such that $\Lambda_e(A) + \varepsilon < \mu$. Let E be the subspace given by Lemma 5.7 and, for $t > 0$, let

$$\Omega_t = \left\{ u \in E : \|u\|_{L^\infty(0,1)} = t \right\}.$$

Then Ω_t has genus $k = \dim E$ and, since $\|\cdot\|_{L^\infty(0,1)}$ and $\|\cdot\|_A$ are equivalent norms on E, there is a constant $C > 0$ such that $\|u\|_A \geq Ct$ for all $u \in \Omega_t$. Now fix $\delta \in (0, 1 - \frac{\Lambda_e(A)+\varepsilon}{\mu})$ and then fix $t \in (0, \pi)$ such that

$$1 - \cos\theta \geq \frac{(1-\delta)}{2}\theta^2 \text{ for all } |\theta| \leq t.$$

Using this and Lemma 5.7, we have that, for $u \in \Omega_t$,

$$\begin{aligned}
j_\mu(u) &= \frac{1}{2}\|u\|_A^2 - \mu \int_0^1 \{1 - \cos u(s)\} ds \\
&\leq \frac{1}{2}\left\{ \|u\|_A^2 - \mu(1-\delta)\int_0^1 u(s)^2 ds \right\} \\
&\leq \frac{1}{2}\|u\|_A^2 \left\{1 - \mu(1-\delta)[\Lambda_e(A)+\varepsilon]^{-1}\right\} \\
&\leq \frac{(Ct)^2}{2}\left\{1 - \mu(1-\delta)[\Lambda_e(A)+\varepsilon]^{-1}\right\} < 0
\end{aligned}$$

and so,

$$0 > \sup_{u \in \Omega_t} j_\mu(u) \geq \inf_{\Omega \in G_k} \sup_{u \in \Omega} j_\mu(u) = b_k.$$

The existence of a sequence $\{u_k\}$ of solutions of Problem P with $j_\mu(u_k) = b_k$ now follows from Theorem 5.3 and Lemma 5.6. Furthermore, $j_\mu(u_k) \to 0$ and $\nabla j_\mu(u_k) = 0$, so the condition (PS) implies that $\{u_k\}$ has a subsequence $\{u_{k_i}\}$ which converges to an element u in H_A. Then, $j_\mu(u) = 0$, $\nabla j_\mu(u) = 0$ and $|u(s)| \leq \pi$ for all $s \in (0,1]$ since u_{k_i} converges to u uniformly on compact subsets of $(0,1]$. By Lemma 5.6 we can conclude that $|u(s)| < \pi$ for all $s \in (0,1]$. But,

$$\begin{aligned}
0 &= 2j_\mu(u) - \langle \nabla j_\mu(u), u \rangle_A \\
&= \mu \int_0^1 \{u(s)\sin u(s) - 2 + 2\cos u(s)\} ds
\end{aligned}$$

and $\theta \sin\theta - 2\{1 - \cos\theta\} < 0$ for $0 < |\theta| < \pi$. This implies that $u \equiv 0$ on $(0,1]$. Since this argument applies to every subsequence of $\{u_k\}$, we can conclude that the whole sequence $\{u_k\}$ converges to 0 in H_A.

Now fix $n \in \mathbb{N}$. We show that there exists $K \in \mathbb{N}$ such that u_k has at least n zeros in $(0,1]$ for all $k \geq K$. First choose $\xi \in (\Lambda_e(A), \mu)$ and any non-trivial solution v of the linearized equation

$$\{A(s)v'(s)\}' + \xi v(s) = 0 \text{ on } (0,1).$$

By Lemma 2.6, there exists $\delta > 0$ such that v has at least $n+1$ zeros in the interval $(\delta, 1]$. Since the sequence $\{u_k\}$ tends to zero in H_A, it converges to 0 uniformly on $[\delta, 1]$ and so there is a constant $K \in \mathbb{N}$ such that

$$q_k(s) = \mu \frac{\sin u_k(s)}{u_k(s)} > \xi \text{ for all } s \in [\delta, 1] \text{ and all } k \geq K.$$

But u_k satisfies the linear equation

$$\{A(s)u'(s)\}' + q_k(s)u(s) = 0 \text{ on } (0,1)$$

and so by the Sturm Comparison Theorem (see Lemma 3.1 in Chapter II of [23]) u_k vanishes at least once between successive zeros of v in $(\delta, 1]$. Hence u_k has at least n zeros in $(\delta, 1]$ for all $k \geq K$.

Remark We have not claimed that for each $k \in \mathbb{N}$ the solution u_k in Theorem 5.8 has exactly k zeros in $(0, 1]$, but, at least under some additional assumptions on A, this is certainly true. One might be able to adapt the arguments in [16] to prove this. The existence of solutions with exactly k zeros should follow from a shooting argument but this will not settle their variational characterization. Let us simply point out one situation where the existence of nodal solutions can be seen in an elementary way. Consider the profile $A(s) = s^2$ and make the change of variables : $t = \sqrt{\mu} \ln s$ and $v(t) = u(s)$. Then the equilibrium equation (1.13) becomes

$$v''(t) + mv'(t) + \sin v(t) = 0 \text{ where } m = \mu^{-1/2}$$

which is the equation of a damped pendulum and the condition

$$\lim_{s \to 0} A(s)u'(s) = 0 \text{ becomes } \lim_{t \to -\infty} e^{mt} v'(t) = 0.$$

The phase portrait is well-known. The point $(\pi, 0)$ is always a saddle but the point $(0,0)$ is a stable spiral when $m < 2$ and a stable node for $m \geq 2$. Fix $m < 2$ and let D denote the branch of the unstable manifold to $(\pi, 0)$ which spirals in to $(0,0)$ as $t \to \infty$. Let $\{(0, d_k) : k \in \mathbb{N}\}$ with $|d_{k+1}| < |d_k|$ be the sequence of points where D crosses the axis $v = 0$. See Figure 15. Now let u_k denote the unique solution of (1.13) with $\mu = 1/m^2$ which satisfies the initial conditions $u_k(1) = 0$ and $u_k'(1) = \sqrt{\mu} d_k$. It is easily seen that u_k is a solution of Problem P which has exactly k zeros in $(0, 1]$. Note the condition $m < 2$ corresponds exactly to the requirement $\mu > \frac{1}{4} = \Lambda(A) = \Lambda_e(A)$ which we expect in this case from the general theory. This example also yields many conclusions about uniqueness and continuous dependence of solutions with a prescribed number of nodes which are not accessible from the general variational approach we have followed.

Fig. 15 Phase portrait for the damped pendulum equation with $m = 0.2$, giving solutions of Problem P with $p = 2$ and $\mu = 25$

6 Buckling of a loaded column

In the Introduction we showed how the discussion of buckling of a tapered column under its own weight could be reduced to the study of Problem P. We now consider another situation which leads directly to Problem P. Here gravity is neglected and a force is applied at the free end, parallel to the direction of clamping at the other end. The problem is easily described using the notation established in the Introduction.

We suppose that the cross-sections $D(z)$ of the column satisfy the conditions (i) to (iv) and we use $r : [0,1] \to \mathbb{R}^2$ and $\theta : [0,1] \to \mathbb{R}$ to describe planar configurations of the curve formed by its centroids. The weight of the column is neglected but a compressive force $F = f(0,-1)$ with $f > 0$ is applied at the free end $r(0)$. The conditions for equilibrium are now

$$\begin{gathered} M'(s) + f\sin\theta(s) = 0 \qquad \text{for } 0 < s < 1, \\ \lim_{s\to 0} M(s) = 0 \text{ and } \theta(1) = 0 \end{gathered}$$

With the Bernoulli - Euler constitutive assumption (1.5), this becomes

$$\begin{gathered} \{I(1-s)\theta'(s)\}' + \mu\sin\theta(s) = 0 \qquad \text{for } 0 < s < 1, \\ \lim_{s\to 0} I(1-s)\theta'(s) = 0 \text{ and } \theta(1) = 0 \end{gathered}$$

where $\mu = \frac{f}{E}$ and the energy is

$$\int_0^1 \frac{1}{2} EI(1-s)\theta'(s)^2 - f\{1-\cos\theta(s)\}\, ds.$$

Putting $A(s) = I(1-s)$, we see that this is precisely Problem P as formulated in the Introduction. In the present situation, the case of a uniform column, where I is constant and so A has tapering of order 0, was completely solved by Euler, see [21] or [28]. If the cross-sections are geometrically similar, then $A(s) = CS(1-s)^2$ and the case $p = 2$ of critical tapering now corresponds to $\lim_{s\to 0} S(1-s)/s = K > 0$.

Remark If gravity is not neglected but acts in the same direction as F, the conditions for equilibrium become

$$\{I(1-s)\theta'(s)\}' + \{\mu + \frac{\rho g}{E}\int_0^s S(1-\tau)d\tau\}\sin\theta(s) = 0 \qquad \text{for } 0 < s < 1,$$
$$\lim_{s\to 0} I(1-s)\theta'(s) = 0 \text{ and } \theta(1) = 0$$

where ρ and g are the density and gravitational constant as in the Introduction. This problem cannot be reduced to the form of Problem P. However, setting $A(s) = I(1-s)$, the differential equation has the form

$$\{A(s)u'(s)\}' + \{\mu + B(s)\}\sin u(s) = 0 \qquad \text{for } 0 < s < 1$$

where $B(s) \to 0$ as $s \to 0$. It turns out that, if the profile A has tapering of order $p \leq 2$, the additional term $B(s)\sin u(s)$ induces a compact, continuously differentiable operator K from the energy space H_A into itself. Thus the problem can be written as

$$u - K(u) - \mu G_A(u) = 0 \text{ for } (\mu, u) \in (0,\infty) \times H_A$$

where G_A is defined by (2.17). We predict that, as far as bifurcation with respect to the load μ is concerned, the situation is qualitatively similar to the case $K = 0$ which we have treated.

Acknowledgement I am grateful to Professor S.J. Cox for drawing my attention to the absence of results about bifurcation for a tapered Euler column.

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